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Nonlinear geometric optics for reflecting uniformly stable pulses

Jean-Francois Coulombel*, Mark Williams†

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Abstract

We provide a justification with rigorous error estimates showing that the leading term in weakly nonlinear geometric optics expansions of highly oscillatory reflecting pulses is close to the uniquely determined exact solution for small wavelengths ε. Pulses reflecting off fixed noncharacteristic boundaries are considered under the assumption that the underlying boundary problem is uniformly spectrally stable in the sense of Kreiss. There are two respects in which these results make rigorous the formal treatment of pulses in Majda and Artola [16], and Hunter, Majda and Rosales [10]. First, we give a rigorous construction of leading pulse profiles in problems where pulses traveling with many distinct group velocities are, unavoidably, present; and second, we provide a rigorous error analysis which yields a rate of convergence of approximate to exact solutions as $\varepsilon \to 0$. Unlike wavetrains, interacting pulses do not produce resonances that affect leading order profiles. However, our error analysis shows the importance of estimating pulse interactions in the construction and estimation of correctors. Our results apply to a general class of systems that includes quasilinear problems like the compressible Euler equations; moreover, the same methods yield a stability result for uniformly stable Euler shocks perturbed by highly oscillatory pulses.

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*CNRS and Université de Nantes, Laboratoire de mathématiques Jean Leray (UMR CNRS 6629), 2 rue de la Houssinière, BP 92208, 44322 Nantes Cedex 3, France. Email: `jean-francois.coulombel@univ-nantes.fr`. Research of J.-F. C. was supported by the French Agence Nationale de la Recherche, contract ANR-08-JCJC-0132-01.

†University of North Carolina, Mathematics Department, CB 3250, Phillips Hall, Chapel Hill, NC 27599. USA. Email: `williams@email.unc.edu`. Research of M.W. was partially supported by NSF grants number DMS-0701201 and DMS-1001616.

1 Introduction

We study highly oscillatory pulse solutions for a general class of hyperbolic equations that includes quasilinear systems like the compressible Euler equations. Our main objective is to construct leading order weakly nonlinear geometric optics expansions of the solutions (which are valuable because, for example, they exhibit important qualitative properties), and to rigorously justify such expansions, that is, to show that they are close in a precise sense to true exact solutions.

A single pulse colliding with a fixed noncharacteristic boundary in an $N \times N$ hyperbolic system will generally give rise to a family of reflected pulses traveling with several distinct group velocities. We study this situation when the underlying boundary problem is assumed to be uniformly spectrally stable in the sense of Kreiss. A formal treatment of this problem was given in Majda-Artola [16], building on an earlier treatment of nonlinear geometric optics for pulses in free space in Hunter-Majda-Rosales [10]. In the papers [16, 10], systems of nonlinear equations for leading order profiles were derived, but their solvability was not discussed. Morever, the questions of the existence of exact solutions on a fixed time interval independent of the wavelength of oscillations (or pulse width) ε, and of the relation between exact and approximate solutions, were not studied there. In this paper, we give a rigorous construction of leading pulse profiles in problems where pulses traveling with many distinct group velocities are, unavoidably, present. In addition, we construct exact solutions on a fixed time interval independent of ε, and provide a rigorous error analysis which yields a rate of convergence of approximate to exact solutions as $\varepsilon \to 0$.

Rigorous treatments of the short-time propagation of a single pulse in free space were given in Alterman-Rauch [1] and Guès-Rauch [8][1]. The methods (e.g., conormal estimates in [1, 8], high-order approximate solutions in [8]) used in the constructions of exact solutions and in the error analyses of these papers do not readily extend to problems involving many pulses with distinct group velocities. The method we use here to construct exact solutions and justify leading term expansions involves replacing the original system (1.1) with an associated singular system (1.3) involving coefficients of order $\frac{1}{\varepsilon}$ and a new unknown $U_\varepsilon(x, \theta_0)$.[2] Exact solutions U_ε to the singular system yield exact solutions to the original system by a substitution

$$u_\varepsilon(x) = U_\varepsilon\left(x, \frac{\phi_0(x')}{\varepsilon}\right),$$

where $\phi_0(x') = x' \cdot \beta$ is the "boundary phase" as in (1.1). Both the singular system and the system of profile equations satisfied by the leading profile $\mathcal{U}^0(x, \theta_0, \xi_d)$ are solved by Picard iteration.

[1] The paper [8] considered "fronts" as well as pulses.

[2] The singular system approach was used in [2] in their study of a single pulse on diffractive time scales.

The error analysis is based on "simultaneous Picard iteration", a method first used in the study of geometric optics for wavetrains in free space in [11]. The idea is to show that for every n, the n-th profile iterate $\mathcal{U}^{0,n}(x,\theta_0,\frac{x_d}{\varepsilon})$ converges as $\varepsilon \to 0$ in an appropriate sense to the n-th exact iterate $U^n_\varepsilon(x,\theta_0)$, and to conclude therefrom that $\mathcal{U}^0(x,\theta_0,\frac{x_d}{\varepsilon})$ is close to $U_\varepsilon(x,\theta_0)$ for ε small. Unlike wavetrains, interacting pulses do not produce resonances that affect leading order profiles. However, our error analysis shows the importance of estimating pulse interactions in the construction and estimation of correctors. Another key tool in the error analysis, discussed further in section 1.4, is the machinery of moment-zero approximations developed in section 4.1. Our use of these approximations was inspired by the "low-frequency cutoff" argument of [2].

The main novelties of this paper are:

1) We give a rigorous treatment of pulses reflecting off boundaries; earlier rigorous work concerned pulses in free space.

2) We provide methods for handling many pulses traveling with distinct group velocities; in particular, we show that although pulse interactions do not produce new pulses at leading order, pulse interactions must be estimated in the construction of correctors and in the error analysis. We distinguish in the estimates between "transversal" and "nontransversal" pulse interactions.

3) In contrast to the treatment of uniformly stable reflecting wavetrains in [6], we are able here to give a rate of convergence of approximate to exact solutions as wavelength $\varepsilon \to 0$.[3]

1.1 Exact solutions and singular systems

In order to study geometric optics for nonlinear problems with highly oscillatory solutions it is important first to settle the question of whether exact solutions exist on a fixed time interval independent of the wavelength (ε in the notation below). A powerful method for studying this problem, introduced in [11] for initial value problems and extended to boundary problems in [24], is to replace the original system with an associated singular system.

On $\overline{\mathbb{R}}^{d+1}_+ = \{x = (x',x_d) = (t,y,x_d) = (t,x'') : x_d \geq 0\}$, consider the $N\times N$ quasilinear hyperbolic boundary problem:

$$\begin{aligned}
&\sum_{j=0}^{d} A_j(v_\varepsilon)\,\partial_{x_j} v_\varepsilon = f(v_\varepsilon)\\
&b(v_\varepsilon)|_{x_d=0} = g_0 + \varepsilon\, G\left(x', \frac{x'\cdot\beta}{\varepsilon}\right) \qquad (1.1)\\
&v_\varepsilon = u_0 \text{ in } t<0,
\end{aligned}$$

where $x_0 = t$ is time, $G(x',\theta_0) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^p)$ decays to zero as $|\theta_0| \to \infty$, with supp $G \subset \{x_0 \geq 0\}$, and the boundary frequency $\beta \in \mathbb{R}^d \setminus \{0\}$[4]. Here the coefficients $A_j \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N^2})$, $f \in C^\infty(\mathbb{R}^N,\mathbb{R}^N)$, and $b \in C^\infty(\mathbb{R}^N,\mathbb{R}^p)$.

Looking for v_ε as a perturbation $v_\varepsilon = u_0 + \varepsilon u_\varepsilon$ of a constant state u_0 such that $f(u_0) = 0$,

[3] In the case of wavetrains there was an "arithmetic obstacle" to obtaining a rate of convergence in the error analysis; namely, the generation of many noncharacteristic, but "almost characteristic", phases by nonlinear interactions. Because pulses interact weakly, that obstacle is absent in the problem studied here.

[4] Wavetrains instead of pulses are obtained by taking $G(x',\theta_0)$ to be periodic in θ_0.

$b(u_0)=g_0$, we obtain for u_ε the system (with slightly different A_j's)

$$
(1.2)\qquad
\begin{aligned}
&(a)\ P(\varepsilon u_\varepsilon,\partial_x)u_\varepsilon := \sum_{j=0}^{d} A_j(\varepsilon u_\varepsilon)\,\partial_{x_j}u_\varepsilon = \mathcal{F}(\varepsilon u_\varepsilon)u_\varepsilon \text{ on } x_d\ge 0\\
&(b)\ B(\varepsilon u_\varepsilon)u_\varepsilon|_{x_d=0} = G\left(x',\frac{x'\cdot\beta}{\varepsilon}\right)\\
&(c)\ u_\varepsilon = 0 \text{ in } t<0,
\end{aligned}
$$

where $B(v)$ is a C^∞ $p\times N$ real matrix defined by

$$b(u_0+\varepsilon u_\varepsilon)=b(u_0)+B(\varepsilon u_\varepsilon)\varepsilon u_\varepsilon$$

and $\mathcal{F}$ is defined similarly. We assume that the boundary $\{x_d=0\}$ is noncharacteristic, that is, $A_d(0)$ is invertible. The other key assumptions, explained in section 1.2, are that $P(v,\partial_x)$ is hyperbolic with characteristics of constant multiplicity for v in a neighborhood of the origin (Assumption 1.1) and that $(P(0,\partial_x),B(0))$ is uniformly stable in the sense of Kreiss (Assumption 1.6).

For any fixed $\varepsilon_0>0$, the standard theory of hyperbolic boundary problems (see e.g., [4, 12]) yields solutions of (1.2) on a fixed time interval $[0,T_{\varepsilon_0}]$ independent of $\varepsilon\ge\varepsilon_0$. However, since Sobolev norms of the boundary data blow up as $\varepsilon\to 0$, the standard theory yields solutions u_ε of (1.2) only on time intervals $[0,T_\varepsilon]$ that shrink to zero as $\varepsilon\to 0$. In section 2, exact (and necessarily unique) solutions to (1.2) of the form $u_\varepsilon(x)=U_\varepsilon(x,\frac{x'\cdot\beta}{\varepsilon})$ are constructed on a time interval independent of $\varepsilon\in(0,\varepsilon_0]$ for ε_0 sufficiently small, where $U_\varepsilon(x,\theta_0)$ satisfies the *singular system* derived by substituting $U_\varepsilon(x,\frac{x'\cdot\beta}{\varepsilon})$ into (1.2):

$$
(1.3)\qquad
\begin{aligned}
&\sum_{j=0}^{d} A_j(\varepsilon U_\varepsilon)\,\partial_{x_j}U_\varepsilon + \frac{1}{\varepsilon}\sum_{j=0}^{d-1} A_j(\varepsilon U_\varepsilon)\,\beta_j\,\partial_{\theta_0}U_\varepsilon = \mathcal{F}(\varepsilon U_\varepsilon)U_\varepsilon,\\
&B(\varepsilon U_\varepsilon)(U_\varepsilon)|_{x_d=0} = G(x',\theta_0),\\
&U_\varepsilon = 0 \text{ in } t<0.
\end{aligned}
$$

As explained in [24], the study of singular systems is greatly complicated by the presence of a boundary. Even if one assumes that the matrices A_j are symmetric (as we do not here), there is no way to obtain an L^2 estimate uniform in ε by a simple integration by parts because of the boundary terms that arise[5]. The blow-up examples of [23] show that, at least in the wavetrain case, for certain boundary frequencies β it is impossible to estimate solutions of (1.3) uniformly with respect to ε in $C(x_d,H^s(x',\theta_0))$ norms, or indeed in *any* norm that dominates the L^∞ norm[6]. We do not know if analogous blow-up examples exist in the pulse case, but it is clear that the proofs of this paper do not apply when β lies in the glancing set (Definition 1.3).

In [7] a class of singular pseudodifferential operators, acting on functions $U(x',\theta_0)$ decaying in θ_0 and having the form

$$
(1.4)\qquad p_s(D_{x',\theta_0})U := \int_{\mathbb{R}^d\times\mathbb{R}} \mathrm{e}^{ix'\xi'+i\theta_0 k}\, p\left(\varepsilon V(x',\theta_0),\xi'+\frac{k\,\beta}{\varepsilon},\gamma\right)\widehat{U}(\xi',k)\,\mathrm{d}\xi'\,\mathrm{d}k,\quad \gamma\ge 1,
$$

[5]The class of symmetric problems with maximal strictly dissipative boundary conditions provides an exception to this statement, but that class is too restrictive for some important applications; for example, the boundary problem that arises in the study of multi-D shocks does not lie in this class.

[6]The problem occurs only for β in the glancing set (Definition 1.3), as the examples of [23] together with the results of [24] show.

was introduced to deal with these difficulties. Observe that after multiplication by $A_d^{-1}(\varepsilon U_\varepsilon)$ and setting $\tilde{A}_j := A_d^{-1}A_j$, $F := A_d^{-1}\mathcal{F}$, (1.3) becomes

$$\begin{aligned}
&\partial_{x_d}U_\varepsilon + \sum_{j=0}^{d-1}\tilde{A}_j(\varepsilon U_\varepsilon)\left(\partial_{x_j} + \frac{\beta_j\partial_{\theta_0}}{\varepsilon}\right)U_\varepsilon \\
&\qquad \equiv \partial_{x_d}U_\varepsilon + \mathbb{A}\left(\varepsilon U_\varepsilon, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right)U_\varepsilon = F(\varepsilon U_\varepsilon)U_\varepsilon, \\
&B(\varepsilon U_\varepsilon)(U_\varepsilon)|_{x_d=0} = G(x',\theta_0), \\
&U_\varepsilon = 0 \text{ in } t<0,
\end{aligned} \tag{1.5}$$

where $\mathbb{A}\left(\varepsilon U_\varepsilon, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right)$ is a (differential) operator that can be expressed in the form (1.4). Kreiss-type symmetrizers $r_s(D_{x',\theta_0})$ in the singular calculus can be constructed for the system (1.5) as in [24] under the assumptions given below. With these one can prove $L^2(x_d, H^s(x',\theta_0))$ estimates uniform in ε for the linearization of (1.5). The main difference with [24] is that we use here a singular pseudodifferential calculus that is specially constructed for pulses in [7]. In the pulse case θ_0 lies in an unbounded set and the exact profile $U_\varepsilon(x,\theta_0)$ has continuous Fourier spectrum. The analysis of [24] relied on a singular calculus for wavetrains. In that case θ_0 lies in S^1 and $U_\varepsilon(x,\theta_0)$ has discrete Fourier spectrum, a fact that was used in several places for proving symbolic calculus rules in [24]. The results of the pulse calculus needed here are recalled in Appendix A.

To progress beyond $L^2(x_d, H^s(x',\theta_0))$ estimates and control L^∞ norms, the boundary frequency β must be restricted to lie in the complement of the glancing set (Definition 1.3). With this extra assumption we are able to use the pulse calculus to block-diagonalize the operator $\mathbb{A}\left(\varepsilon U_\varepsilon, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right)$ and thereby prove estimates uniform with respect to ε in the spaces

$$E^s_T = C(x_d, H^s_T(x',\theta_0)) \cap L^2(x_d, H^{s+1}_T(x',\theta_0)). \tag{1.6}$$

These spaces are algebras and are contained in L^∞ for $s > \frac{d+1}{2}$. For large enough s, as determined by the requirements of the calculus, existence of solutions to (A.6) in E^s_T on a time interval $[0,T]$ independent of $\varepsilon \in (0,\varepsilon_0]$ follows by Picard iteration (see Theorem 1.12).

1.2 Assumptions and main results

Before continuing with an overview of the strategies for constructing profiles and for showing that approximate solutions are close to exact solutions, we now give a precise statement of our assumptions and main results.

We make the following hyperbolicity assumption on the system (1.2):

Assumption 1.1. *The matrix $A_0 = I$. For an open neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^N$, there exists an integer $q \geq 1$, some real functions $\lambda_1,\dots,\lambda_q$ that are C^∞ on $\mathcal{U} \times \mathbb{R}^d \setminus \{0\}$ and homogeneous of degree 1 and analytic in ξ, and there exist some positive integers $\nu_1,\dots,\nu_q$ such that:*

$$\det\left[\tau I + \sum_{j=1}^d \xi_j A_j(u)\right] = \prod_{k=1}^q \left(\tau + \lambda_k(u,\xi)\right)^{\nu_k}$$

for $u \in \mathcal{U}$ and $\xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d \setminus \{0\}$. Moreover the eigenvalues $\lambda_1(u,\xi),\dots,\lambda_q(u,\xi)$ are semi-simple (their algebraic multiplicity equals their geometric multiplicity) and satisfy $\lambda_1(u,\xi) < \cdots < \lambda_q(u,\xi)$ for all $u \in \mathcal{U}$, $\xi \in \mathbb{R}^d \setminus \{0\}$.

We restrict our analysis to noncharacteristic boundaries and therefore make the following:

Assumption 1.2. *For $u \in \mathcal{U}$ the matrix $A_d(u)$ is invertible and the matrix $B(u)$ has maximal rank, its rank p being equal to the number of positive eigenvalues of $A_d(u)$ (counted with their multiplicity).*

In the normal modes analysis for the linearization of (1.2) at $0 \in \mathcal{U}$, one first performs a Laplace transform in the time variable t and a Fourier transform in the tangential space variables y. We let $\tau - i\,\gamma \in \mathbb{C}$ and $\eta \in \mathbb{R}^{d-1}$ denote the dual variables of t and y. We introduce the symbol

$$\mathcal{A}(\zeta) := -i\,A_d^{-1}(0)\left((\tau - i\gamma)\,I + \sum_{j=1}^{d-1} \eta_j\,A_j(0)\right)\,, \quad \zeta := (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}\,.$$

For future use, we also define the following sets of frequencies:

$$\Xi := \Big\{(\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus (0,0) : \gamma \geq 0\Big\}\,, \qquad \Sigma := \Big\{\zeta \in \Xi : \tau^2 + \gamma^2 + |\eta|^2 = 1\Big\}\,,$$
$$\Xi_0 := \Big\{(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus (0,0)\Big\} = \Xi \cap \{\gamma = 0\}\,, \qquad \Sigma_0 := \Sigma \cap \Xi_0\,.$$

Henceforth we suppress the u in $\lambda_k(u,\xi)$ when it is evaluated at $u = 0$ and write $\lambda_k(0,\xi) = \lambda_k(\xi)$. Two key objects in our analysis are the hyperbolic region and the glancing set that are defined as follows:

Definition 1.3.
- *The hyperbolic region $\mathcal{H}$ is the set of all $(\tau,\eta) \in \Xi_0$ such that the matrix $\mathcal{A}(\tau,\eta)$ is diagonalizable with purely imaginary eigenvalues.*
- *Let G denote the set of all $(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^d$ such that $\xi \neq 0$ and there exists an integer $k \in \{1,\dots,q\}$ satisfying:*
$$\tau + \lambda_k(\xi) = \frac{\partial \lambda_k}{\partial \xi_d}(\xi) = 0\,.$$
If $\pi(G)$ denotes the projection of G on the d first coordinates (in other words $\pi(\tau,\xi) = (\tau,\xi_1,\dots,\xi_{d-1})$ for all (τ,ξ)), the glancing set $\mathcal{G}$ is $\mathcal{G} := \pi(G) \subset \Xi_0$.

We recall the following result that is due to Kreiss [12] in the strictly hyperbolic case (when all integers ν_j in Assumption 1.1 equal 1) and to Métivier [19] in our more general framework:

Proposition 1.4 ([12, 19])**.** *Let Assumptions 1.1 and 1.2 be satisfied. Then for all $\zeta \in \Xi \setminus \Xi_0$, the matrix $\mathcal{A}(\zeta)$ has no purely imaginary eigenvalue and its stable subspace $\mathbb{E}^s(\zeta)$ has dimension p.*[7] *Furthermore, $\mathbb{E}^s$ defines an analytic vector bundle over $\Xi \setminus \Xi_0$ that can be extended as a continuous vector bundle over Ξ.*

For all $(\tau,\eta) \in \Xi_0$, we let $\mathbb{E}^s(\tau,\eta)$ denote the continuous extension of $\mathbb{E}^s$ to the point (τ,η). The analysis in [19] shows that away from the glancing set $\mathcal{G} \subset \Xi_0$, $\mathbb{E}^s(\zeta)$ depends analytically on ζ, and the hyperbolic region $\mathcal{H}$ does not contain any glancing point.

Next we define the hyperbolic operator

$$L(\partial_x) := \partial_t + \sum_{j=1}^{d} A_j(0)\partial_{x_j}$$

and recall the definition of uniform stability [12, 4]:

[7] The stable subspace is the direct sum of the generalized eigenspaces associated to eigenvalues with negative real part.

Definition 1.5. *The problem* (1.2) *is said to be* uniformly stable *at* $u = 0$ *if the linearized operators* $(L(\partial_x), B(0))$ *at* $u = 0$ *are such that*

$$B(0) : \mathbb{E}^s(\tau - i\gamma, \eta) \to \mathbb{C}^p \text{ is an isomorphism for all } (\tau - i\gamma, \eta) \in \Sigma.$$

Assumption 1.6. *The problem* (1.2) *is* uniformly stable *at* $u = 0$.

It is clear that uniform stability at $u = 0$ implies uniform stability at nearby states (and therefore at all $u \in \mathcal{U}$ up to restricting $\mathcal{U}$). Thus, there is a slight redundancy in Assumptions 1.2 and 1.6 as far as the rank of $B(0)$ is concerned.

Boundary and interior phases. We consider a planar real phase ϕ_0 defined on the boundary:

$$\phi_0(t, y) := \underline{\tau}\, t + \underline{\eta} \cdot y \,, \quad (\underline{\tau}, \underline{\eta}) \in \Xi_0 \,. \tag{1.7}$$

As follows from earlier works (e.g. [16]), oscillations on the boundary associated with the phase ϕ_0 give rise to oscillations in the interior associated with some planar phases ϕ_m. These phases are characteristic for the hyperbolic operator $L(\partial_x)$ and their trace on the boundary equals ϕ_0. For now we make the following:

Assumption 1.7. *The phase* ϕ_0 *defined by* (1.7) *satisfies* $(\underline{\tau}, \underline{\eta}) \in \mathcal{H}$.

Thanks to Assumption 1.7, we know that the matrix $\mathcal{A}(\underline{\tau}, \underline{\eta})$ is diagonalizable with purely imaginary eigenvalues. These eigenvalues are denoted $i\,\omega_1, \dots, i\,\omega_M$, where the ω_m's are real and pairwise distinct. The ω_m's are the roots (and all the roots are real) of the dispersion relation:

$$\det \Big[\underline{\tau}\, I + \sum_{j=1}^{d-1} \underline{\eta}_j\, A_j(0) + \omega\, A_d(0) \Big] = 0 \,.$$

To each root ω_m there corresponds a unique integer $k_m \in \{1, \dots, q\}$ such that $\underline{\tau} + \lambda_{k_m}(\underline{\eta}, \omega_m) = 0$. We can then define the following real[8] phases and their associated group velocities:

$$\forall\, m = 1, \dots, M \,, \quad \phi_m(x) := \phi_0(t, y) + \omega_m\, x_d \,, \quad \mathbf{v}_m := \nabla \lambda_{k_m}(\underline{\eta}, \omega_m) \,. \tag{1.8}$$

Let us observe that each group velocity $\mathbf{v}_m$ is either incoming or outgoing with respect to the space domain $\mathbb{R}^d_+$: the last coordinate of $\mathbf{v}_m$ is nonzero. This property holds because $(\underline{\tau}, \underline{\eta})$ does not belong to the glancing set $\mathcal{G}$. We can therefore adopt the following classification:

Definition 1.8. *The phase* ϕ_m *is said to be incoming if the group velocity* $\mathbf{v}_m$ *is incoming (that is,* $\partial_{\xi_d} \lambda_{k_m}(\underline{\beta}, \omega_m) > 0$*), and outgoing if the group velocity* $\mathbf{v}_m$ *is outgoing* ($\partial_{\xi_d} \lambda_{k_m}(\underline{\beta}, \omega_m) < 0$).

In all that follows, we let $\mathcal{I}$ denote the set of indices $m \in \{1, \dots, M\}$ such that ϕ_m is an incoming phase, and $\mathcal{O}$ denote the set of indices $m \in \{1, \dots, M\}$ such that ϕ_m is an outgoing phase. If $p \geq 1$, then $\mathcal{I}$ is nonempty, while if $p \leq N - 1$, $\mathcal{O}$ is nonempty (this follows from Lemma 1.9 below).

Main results. We will use the notation:

$$L(\tau, \xi) := \tau\, I + \sum_{j=1}^{d} \xi_j\, A_j(0) \,,$$
$$\beta = (\underline{\tau}, \underline{\eta}),\ x' = (t, y),\ \phi_0(x') = \beta \cdot x'.$$

[8] If $(\underline{\tau}, \underline{\eta})$ does not belong to the hyperbolic region $\mathcal{H}$, some of the phases ϕ_m may be complex, see e.g. [21, 23, 14, 18, 9]. Moreover, glancing phases introduce a new scale $\sqrt{\varepsilon}$ as well as boundary layers.

For each phase ϕ_m, $\mathrm{d}\phi_m$ denotes the differential of the function ϕ_m with respect to its argument $x = (t, y, x_d)$. It follows from Assumption 1.1 that the eigenspace of $\mathcal{A}(\beta)$ associated with the eigenvalue $i\,\underline{\omega}_m$ coincides with the kernel of $L(\mathrm{d}\phi_m)$ and has dimension ν_{k_m}. The following well-known lemma, whose proof is recalled in [5], gives a useful decomposition of $\mathbb{E}^s$ in the hyperbolic region.

Lemma 1.9. *The stable subspace* $\mathbb{E}^s(\underline{\tau}, \underline{\eta})$ *admits the decomposition:*

$$\mathbb{E}^s(\underline{\tau}, \underline{\eta}) = \oplus_{m\in\mathcal{I}} \operatorname{Ker} L(\mathrm{d}\phi_m)\,, \tag{1.9}$$

and each vector space in the decomposition (1.9) *admits a basis of real vectors.*

The next Lemma, also proved in [5], gives a useful decomposition of $\mathbb{C}^N$ and introduces projectors needed later for formulating and solving the profile equations.

Lemma 1.10. *The space* $\mathbb{C}^N$ *admits the decomposition:*

$$\mathbb{C}^N = \oplus_{m=1}^M \operatorname{Ker} L(\mathrm{d}\phi_m) \tag{1.10}$$

and each vector space in (1.10) *admits a basis of real vectors. If we let* $P_1, \dots, P_M$ *denote the projectors associated with the decomposition* (1.10)*, then there holds* $\operatorname{Im} A_d^{-1}(0)\, L(\mathrm{d}\phi_m) = \operatorname{Ker} P_m$ *for all* $m = 1, \dots, M$*.*

For each $m \in \{1, \dots, M\}$ we let

$$r_{m,k},\ k = 1, \dots, \nu_{k_m}$$

denote a basis of $\ker L(\mathrm{d}\phi_m)$ consisting of real vectors. In section 3, we construct an approximate solution u^a_ε of (1.2) of the form

$$u^a_\varepsilon(x) = \sum_{m\in\mathcal{I}} \sum_{k=1}^{\nu_{k_m}} \sigma_{m,k}\left(x, \frac{\phi_m}{\varepsilon}\right) r_{m,k}, \tag{1.11}$$

where the $\sigma_{m,k}(x, \theta_m)$ are C^1 functions decaying to zero as $|\theta_m| \to \infty$, which describe the propagation of pulses with group velocity $\mathbf{v}_m$ (see Proposition 3.6). Observe that if one plugs the expression (1.11) of u^a_ε into $P(\varepsilon u_\varepsilon, \partial_x)u_\varepsilon$, the terms of order $1/\varepsilon$ vanish, leaving an $O(1)$ error, *regardless* of how the $\sigma_{m,k}$ are chosen. The interior profile equations satisfied by these functions are solvability conditions that permit this $O(1)$ error to be (at least partially) removed by a corrector that is sublinear (in fact bounded) as $|\theta_m| \to \infty$. Additional conditions on the profiles come, of course, from the boundary conditions.

For use in the remainder of the introduction and later, we collect some notation here.

Notations 1.11. *(a) Let* $\Omega := \overline{\mathbb{R}}^{d+1}_+ \times \mathbb{R}^1$, $\Omega_T := \Omega \cap \{-\infty < t < T\}$, $b\Omega := \mathbb{R}^d \times \mathbb{R}^1$, $b\Omega_T := b\Omega \cap \{-\infty < t < T\}$*, and set* $\omega_T := \overline{\mathbb{R}}^{d+1}_+ \cap \{-\infty < t < T\}$*.*

(b) For $s \geq 0$ *let* $H^s \equiv H^s(b\Omega)$*, the standard Sobolev space with norm* $\langle V(x', \theta_0)\rangle_s$*.*

(c) $L^2H^s \equiv L^2(x_d, H^s(b\Omega))$ *with* $|U(x, \theta_0)|_{L^2H^s} \equiv |U|_{0,s}$*.*

(d) $CH^s \equiv C(x_d, H^s(b\Omega))$ *with* $|U(x, \theta_0)|_{CH^s} \equiv \sup_{x_d\geq 0} |U(., x_d, .)|_{H^s} \equiv |U|_{\infty,s}$ *(note that* $CH^s \subset L^\infty H^s$*).*

(e) $C^{0,M}(\mathbb{R}^{d+1}_+ \times \mathbb{R}) \equiv \{V(x', x_d, \theta_0) \in C\left(\mathbb{R}_+, C^M_b(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^N)\right)\}$ *where* C^M_b *denotes the space of* M *times differentiable functions with derivatives up to the order* M *bounded.*

(f) Similarly, $H^s_T \equiv H^s(b\Omega_T)$ with norm $\langle V\rangle_{s,T}$ and $L^2H^s_T \equiv L^2(x_d, H^s_T)$, $CH^s_T \equiv C(x_d, H^s_T)$ have norms $|U|_{0,s,T}$, $|U|_{\infty,s,T}$ respectively.

(g)When the domains of x_d and (x', θ_0) are clear, we sometimes use the self-explanatory notation $C(x_d, H^s(x', \theta_0))$ or $L^2(x_d, H^s(x', \theta_0))$.

(h) For $r \geq 0$, $[r]$ is the smallest integer $\geq r$.

(i) $M_0 := 3\,d + 5$,

The main result of section 2 is the following theorem, which gives the existence of exact solutions to the singular system (1.3), or equivalently (1.5), and the original system (1.2) on a time interval independent of the wavelength ε:

Theorem 1.12. *Under Assumptions 1.1, 1.2, 1.6, 1.7, consider the quasilinear boundary problem (1.2), where $G(x', \theta_0) \in H^{s+1}(b\Omega)$, $s \geq [M_0 + \frac{d+1}{2}]$, satisfies*

$$\mathrm{Supp}\; G \subset \{t \geq 0\}\,.$$

There exist $\varepsilon_0 > 0$, $T_0 > 0$ independent of $\varepsilon \in (0, \varepsilon_0]$, and a unique $U_\varepsilon(x, \theta_0) \in CH^s_{T_0} \cap L^2H^{s+1}_{T_0}$ satisfying the singular problem (1.5), so that

$$u_\varepsilon(x) := U_\varepsilon\left(x, \frac{x' \cdot \beta}{\varepsilon}\right)\,,$$

is the unique C^1 solution of (1.2) on ω_{T_0}.

Remark 1.13. The regularity requirement $s \geq [M_0 + \frac{d+1}{2}]$ in the above theorem is needed in order to apply the singular pseudodifferential calculus introduced in [7].

We can now state the main result of this paper. This theorem is actually a corollary of the result for singular systems given in Theorem 4.16.

Theorem 1.14. *Under the same assumptions as in Theorem 1.12, there exists $T_0 > 0$ and functions $\sigma_{m,k}(x, \theta_m) \in C^1(\Omega_{T_0})$ satisfying the leading order profile equations (5.3) and defining an approximate solution u^a_ε as in (1.11) such that*

$$\lim_{\varepsilon \to 0} u_\varepsilon - u^a_\varepsilon = 0 \text{ in } L^\infty(\omega_{T_0}),$$

where $u_\varepsilon \in C^1(\omega_{T_0})$ is the unique exact solution of (1.2). In fact we obtain the rate of convergence

$$|u_\varepsilon - u^a_\varepsilon|_{L^\infty(\omega_{T_0})} \leq C\,\varepsilon^{\frac{1}{2M_1+5}}, \text{ where } M_1 := \left[\frac{d}{2} + 3\right].$$

Theorem 1.14 can be recast in a form where the pulses originate in initial data at $t = 0$ and reflect off the boundary $\{x_d = 0\}$. This requires a discussion similar to that given in section 3.2 of [6] to justify the reduction of the initial boundary value problem with data prescribed at $t = 0$ to a forward boundary problem (with data identically zero in $t < 0$), so we omit that discussion here.

1.3 Profile equations

In d space variables (x'', x_d), consider the quasilinear problem equivalent to (1.2)

$$\begin{aligned}
&\partial_d u_\varepsilon + \sum_{j=0}^{d-1} \tilde{A}_j(\varepsilon u_\varepsilon)\partial_j u_\varepsilon = F(\varepsilon u_\varepsilon)u_\varepsilon \text{ in } x_d \geq 0\\
&B(\varepsilon u_\varepsilon)u_\varepsilon = G(x', \theta_0)|_{\theta_0 = \frac{\phi_0}{\varepsilon}} \text{ on } x_d = 0\\
&u_\varepsilon = 0 \text{ in } t < 0,
\end{aligned}$$

where $G(x', \theta_0)$ decays to 0 like $\langle\theta_0\rangle^{-k}$ (for some $k \geq 2$ to be specified later) as $|\theta_0| \to \infty$. For ease of exposition we will begin by considering the 3×3 case, which contains all the main difficulties. In section 5, we describe the changes needed to treat the general case. We define the boundary phase $\phi_0 := \beta \cdot x'$, the real eigenvalues ω_m of $\mathcal{A}(\beta)$, and the phases $\phi_m := \phi_0 + \omega_m \, x_d$ as in (1.8), where $\beta \in \mathcal{H}$. We assume that the eigenvalues ω_m are pairwise distinct. For the sake of clarity, we also assume that ω_1 and ω_3 are incoming (or causal) and ω_2 is outgoing. (The same kind of arguments would apply if two of the phases were outgoing and only one was incoming.) The corresponding right and left eigenvectors of the real matrix $-i\,\mathcal{A}(\beta)$ are denoted r_j and l_j, $j = 1, 2, 3$.

Below we frequently suppress ε-dependence in the notation. For functions $\mathcal{U}(x, \theta_0, \xi_d)$ and $\mathcal{V}(x, \theta_0, \xi_d)$, define

$$\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) := \partial_{\xi_d} + \sum_{j=0}^{d-1} \beta_j \, \tilde{A}_j(0) \, \partial_{\theta_0} = \partial_{\xi_d} + \tilde{A}(\beta) \, \partial_{\theta_0} \text{ and } \tilde{L}(\partial) := \partial_d + \sum_{j=0}^{d-1} \tilde{A}_j(0) \, \partial_j \,,$$

$$M(\mathcal{U}, \partial_{\theta_0} \mathcal{V}) := \sum_{j=0}^{d-1} \beta_j \, (\mathrm{d}\tilde{A}_j(0) \cdot \mathcal{U}) \, \partial_{\theta_0} \mathcal{V} \,.$$

Formally looking for a corrected approximate solution of the form

$$u_\varepsilon^c(x) = \left[\mathcal{U}^0(x, \theta_0, \xi_d) + \varepsilon \, \mathcal{U}^1(x, \theta_0, \xi_d)\right]|_{\theta_0 = \frac{\phi_0}{\varepsilon}, \xi_d = \frac{x_d}{\varepsilon}} \,,$$

we obtain interior profile equations

$$\begin{aligned} &\varepsilon^{-1}: \quad \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \, \mathcal{U}^0 = 0 \,, \\ &\varepsilon^0: \quad \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \, \mathcal{U}^1 + \tilde{L}(\partial) \, \mathcal{U}^0 + M(\mathcal{U}^0, \partial_{\theta_0} \mathcal{U}^0) = F(0) \, \mathcal{U}^0 \,, \end{aligned} \tag{1.12}$$

and the boundary equation

$$\varepsilon^0 : B(0) \, \mathcal{U}^0|_{x_d=0, \xi_d=0} = G(x', \theta_0) \,. \tag{1.13}$$

Consider the first equation in (1.12). A function $\mathcal{U}^0(x, \theta_0, \xi_d)$, taking values in $\mathbb{R}^3$ and assumed to be C^1 for the moment, can always be written

$$\mathcal{U}^0 = \tilde{\sigma}_1(x, \theta_0, \xi_d) \, r_1 + \tilde{\sigma}_2(x, \theta_0, \xi_d) \, r_2 + \tilde{\sigma}_3(x, \theta_0, \xi_d) \, r_3 \,.$$

Using the matrix $[r_1 \; r_2 \; r_3]$ to diagonalize $\tilde{A}(\beta)$, we find that the scalar $\tilde{\sigma}_i$ must satisfy

$$(\partial_{\xi_d} - \omega_i \, \partial_{\theta_0}) \, \tilde{\sigma}_i = 0, \;\; i = 1, 2, 3, \;\; \text{in } \{(x, \theta_0, \xi_d) : \theta_0 \in \mathbb{R}, \xi_d \geq 0\}.$$

This implies that the $\tilde{\sigma}_i$'s have the form

$$\tilde{\sigma}_i(x, \theta_0, \xi_d) = \sigma_i(x, \theta_0 + \omega_i \xi_d) \text{ for some } \sigma_i(x, \theta_i).$$

Using (1.13), we find

$$B(0) \left(\sum_{i=1,3} \sigma_i(x', 0, \theta_0) \, r_i \right) = G(x', \theta_0) - B(0)(\sigma_2(x', 0, \theta_0) \, r_2), \tag{1.14}$$

Remark 1.15. 1. We expect the $\sigma_i(x,\theta_i)$ to decay polynomially to 0 as $|\theta_i|\to\infty$. To prove this we must formulate and solve profile equations for the σ_i's. For this we use an approach inspired by the formal constructions in [10] and [17].

2. Instead of $\sigma_i(x,\theta_i)$ we shall sometimes write $\sigma_i(x,\theta)$ with the understanding that θ is a placeholder for $\theta_0+\omega_i\,\xi_d$ when it appears as an argument of σ_i.

To get transport equations for the σ_i's, we consider (1.12) (ε^0):

$$\tilde{\mathcal{L}}(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U}^1=-\Big(\tilde{L}(\partial)\mathcal{U}^0+M(\mathcal{U}^0,\partial_{\theta_0}\mathcal{U}^0)\Big)+F(0)\mathcal{U}^0:=\mathcal{F}(x,\theta_0,\xi_d). \tag{1.15}$$

The corrector $\mathcal{U}^1$ can be written as

$$\mathcal{U}^1=t_1(x,\theta_0,\xi_d)r_1+t_2(x,\theta_0,\xi_d)r_2+t_3(x,\theta_0,\xi_d)r_3.$$

Diagonalizing again we find that the t_i's must satisfy

$$(\partial_{\xi_d}-\omega_i\partial_{\theta_0})t_i(x,\theta_0,\xi_d)=l_i\cdot\mathcal{F}:=\mathcal{F}_i(x,\theta_0,\xi_d),\ i=1,2,3. \tag{1.16}$$

The general solution to (1.16) is

$$t_i(x,\theta_0,\xi_d)=\tau_i^*(x,\theta_0+\omega_i\xi_d)+\int_0^{\xi_d}\mathcal{F}_i(x,\theta_0+\omega_i(\xi_d-s),s)\,\mathrm{d}s, \tag{1.17}$$

where τ_i^* is arbitrary. This can be rewritten

$$\begin{aligned}&t_i(x,\theta_0,\xi_d)\\&=\tau_i^*(x,\theta_0+\omega_i\xi_d)+\int_0^{\infty}\mathcal{F}_i(x,\theta_0+\omega_i(\xi_d-s),s)\,\mathrm{d}s+\int_\infty^{\xi_d}\mathcal{F}_i(x,\theta_0+\omega_j(\xi_d-s),s)\,\mathrm{d}s\\&=\tau_i(x,\theta_0+\omega_i\xi_d)+\int_\infty^{\xi_d}\mathcal{F}_i(x,\theta_0+\omega_i(\xi_d-s),s)\,\mathrm{d}s\,,\end{aligned} \tag{1.18}$$

provided the integrals in (1.18) exist.

We will need the following modification of a classical lemma due to Lax [13]. We refer to [6, Lemma 2.11] for the proof.

Proposition 1.16. *Let $W(x,\theta_0,\xi_d)=\sum_{i=1}^3 w_i(x,\theta_0,\xi_d)r_i$ be any C^1 function. Then*

$$\tilde{L}(\partial)W=\sum_{i=1}^3(X_{\phi_i}w_i)\,r_i+\sum_{i=1}^3\big(\sum_{k\neq i}V_k^iw_k\big)\,r_i\,,$$

where X_{ϕ_i} is the characteristic vector field[9]

$$X_{\phi_i}:=\partial_{x_d}+\sum_{j=0}^{d-1}-\partial_{\xi_j}\omega_i(\beta)\partial_{x_j}\,,$$

and V_k^i for $k\neq i$ is the tangential vector field

$$V_k^i:=\sum_{l=0}^{d-1}(l_i\,\tilde{A}_l(0)\,r_k)\,\partial_{x_l}.$$

[9]This vector field is a scalar multiple of $\partial_t+\nabla\lambda_{k_i}(\underline{\eta},\omega_i)\cdot\nabla_{x''}$ that describes propagation at the group velocity $\mathbf{v}_i$; see (1.8).

We see from (1.15) that $\mathcal{F}_i(x,\theta_0,\xi_d)$ has the form

$$\mathcal{F}_i(x,\theta_0,\xi_d) = -X_{\phi_i}\tilde{\sigma}_i - \sum_k c^i_k\,\tilde{\sigma}_k\,\partial_{\theta_0}\tilde{\sigma}_k - \sum_{l\neq m} d^i_{l,m}\,\tilde{\sigma}_l\,\partial_{\theta_0}\tilde{\sigma}_m + \sum_k e^i_k\,\tilde{\sigma}_k - \sum_{k\neq i} V^i_k\,\tilde{\sigma}_k\,, \tag{1.19}$$

where we recall $\tilde{\sigma}_p(x,\theta_0,\xi_d) = \sigma_p(x,\theta_0+\omega_p\theta_d)$. The coefficients in (1.19) are defined by[10]

$$c^i_k := l_i\sum_{j=0}^{d-1}\beta_j\,(\mathrm{d}\tilde{A}_j(0)\cdot r_k)\,r_k\,,\quad d^i_{l,m} := l_i\sum_{j=0}^{d-1}\beta_j\,(\mathrm{d}\tilde{A}_j(0)\cdot r_l)\,r_m\,,\quad e^i_k := l_i\,F(0)\,r_k\,.$$

Thus, we compute

$$\begin{aligned}
&\mathcal{F}_i(x,\theta_0+\omega_i(\xi_d-s),s)\\
&= -(X_{\phi_i}\sigma_i + c^i_i\,\sigma_i\,\partial_\theta\sigma_i - e^i_i\,\sigma_i)(x,\theta_0+\omega_i\xi_d)\\
&\quad -\sum_{k\neq i} c^i_k\,\sigma_k(x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))\,\partial_\theta\sigma_k(x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))\\
&\quad -\sum_{m\neq i} d^i_{i,m}\,\sigma_i(x,\theta_0+\omega_i\xi_d)\,\partial_\theta\sigma_m(x,\theta_0+\omega_i\xi_d+s(\omega_m-\omega_i))\\
&\quad -\sum_{l\neq i} d^i_{l,i}\,\sigma_l(x,\theta_0+\omega_i\xi_d+s(\omega_l-\omega_i))\,\partial_\theta\sigma_i(x,\theta_0+\omega_i\xi_d)\\
&\quad -\sum_{l\neq m,l\neq i,m\neq i} d^i_{l,m}\,\sigma_l(x,\theta_0+\omega_i\xi_d+s(\omega_l-\omega_i))\,\partial_\theta\sigma_m(x,\theta_0+\omega_i\xi_d+s(\omega_m-\omega_i))\\
&\quad +\sum_{k\neq i}(e^i_k-V^i_k)\,\sigma_k(x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))\,.
\end{aligned} \tag{1.20}$$

We look for functions $\sigma_i(x,\theta)$ that decay at least at the rate $\langle\theta\rangle^{-2}$. So we assume now and verify later that they have this property. Then the integral

$$\int_0^{\xi_d}\mathcal{F}_i(x,\theta_0+\omega_i(\xi_d-s),s)\,\mathrm{d}s \tag{1.21}$$

is sublinear in (θ_0,ξ_d) (a condition that must be satisfied by $\mathcal{U}^1$ if $\varepsilon\mathcal{U}^1$ is to make sense as a corrector) if and only if the sum of the first three terms on the right in (1.20) is 0. In that case the integral (1.21) is actually bounded, since the remaining terms in (1.20) have good decay in s. This sublinearity condition gives the profile equations for the σ_i's:

$$\begin{aligned}
&X_{\phi_i}\sigma_i + c^i_i\,\sigma_i\,\partial_{\theta_i}\sigma_i - e^i_i\,\sigma_i = 0,\ i=1,2,3\\
&(\sigma_i(x',0,\theta_0), i=1,3) = \mathcal{B}\,\big(G(x',\theta_0),\sigma_2(x',0,\theta_0)\big)\,,\\
&\sigma_i = 0 \text{ in } t<0.
\end{aligned} \tag{1.22}$$

where $\mathcal{B}$ is a well-determined linear function of its arguments whose existence is given by Lemma 1.9 and the uniform stability assumption (the matrix $[B(0)\,r_1\ B(0)\,r_3]$ in (1.14) is invertible). As expected from the general rule of thumb, pulses of different families do not interact at the leading order, meaning that the evolution equations for the amplitudes σ_i's are decoupled. In Proposition 3.6, we show that system (1.22) is uniquely solvable on some time interval $[0,T_1]$, that $\sigma_2=0$ and that σ_i, $i=1,3$, decay at the rate $\langle\theta\rangle^{-k}$ for some $k\geq 2$ to be determined.

[10] We refer to section 5 for the general case.

Remark 1.17. The equations (1.22) and our assumption that the σ_i decay at least at the rate $\langle\theta\rangle^{-2}$ imply that the integrals in (1.18) all do exist. This argument is made more precise below.

Next we introduce an averaging operator $\mathbf{E}$ and a solution operator $\mathbf{R}_\infty$ that will be useful in the error analysis of the next paragraph. Motivated by the form of $\mathcal{F}_i$ in (1.19), we make the following definition.

Definition 1.18 (Type $\mathcal{F}$ functions)**.** *Suppose*

$$F(x,\theta_0,\xi_d) = \sum_{i=1}^{3} F_i(x,\theta_0,\xi_d)\, r_i, \tag{1.23}$$

where each F_i has the form

$$F_i(x,\theta_0,\xi_d) = \sum_{k=1}^{3} f_k^i(x,\theta_0+\omega_k\,\xi_d) + \sum_{l\le m=1}^{3} g_{l,m}^i(x,\theta_0+\omega_l\,\xi_d)\, h_{l,m}^i(x,\theta_0+\omega_m\,\xi_d), \tag{1.24}$$

where the functions $f_k^i(x,\theta)$, $g_{l,m}^i(x,\theta)$, $h_{l,m}^i(x,\theta)$ are real-valued, C^1, and decay along with their first order partials at the rate $O(\langle\theta\rangle^{-2})$ uniformly with respect to x. We then say that F is of type $\mathcal{F}$*. For such F define*

$$\mathbf{E}F(x,\theta_0,\xi_d) := \sum_{j=1}^{3}\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T l_j\cdot F(x,\theta_0+\omega_j\,(\xi_d-s),s)\,\mathrm{d}s\right) r_j\,.$$

Remark 1.19. 1.) For F as in (1.23)-(1.24), we have

$$\mathbf{E}F = \sum_{i=1}^{3}\tilde{F}_i(x,\theta_0+\omega_i\,\xi_d)\, r_i,\ \text{ where } \tilde{F}_i(x,\theta) := f_i^i(x,\theta) + g_{i,i}^i(x,\theta)\, h_{i,i}^i(x,\theta)\,. \tag{1.25}$$

2.) Observe that $\mathcal{F}$ as defined in (1.15) is of type $\mathcal{F}$ (hence the terminology), provided the σ_i's have sufficiently regularity and decay in θ. In that case, we obtain

$$\mathbf{E}\mathcal{F}(x,\theta_0,\xi_d) = -\sum_{i=1}^{3}\left(X_{\phi_i}\sigma_i + c_i^i\,\sigma_i\,\partial_{\theta_i}\sigma_i - e_i^i\,\sigma_i\right) r_i\,,\qquad \text{where } \sigma_i = \sigma_i(x,\theta_0+\omega_i\,\xi_d).$$

Remark 1.20. The definition of $\mathbf{E}$ can be extended to more general functions. For example, if

$$F = \sum_{i=1}^{3} F_i(x,\theta_0+\omega_i\,\xi_d)\, r_i,$$

where the $F_i(x,\theta)$ are arbitrary continuous functions, the limits that define $\mathbf{E}F$ exist and we have $\mathbf{E}F = F$. For another example, suppose F is of type $\mathcal{F}$ and satisfies $EF = 0$. Define

$$\mathbf{R}_\infty F(x,\theta_0,\xi_d) := \sum_{i=1}^{3}\left(\int_\infty^{\xi_d} F_i(x,\theta_0+\omega_i(\xi_d-s),s)\,\mathrm{d}s\right) r_i\,. \tag{1.26}$$

Then the limits defining $\mathbf{R}_\infty F$ and $\mathbf{E}\mathbf{R}_\infty F$ exist and we have $\mathbf{E}\mathbf{R}_\infty F = 0$.

Proposition 1.21. *Suppose F is of type $\mathcal{F}$ and satisfies $\mathbf{E}F = 0$. Then $\mathbf{R}_\infty F$ is bounded and*

$$\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})\,\mathbf{R}_\infty F = \mathbf{R}_\infty\,\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})F = F = (I - \mathbf{E})F.$$

Proof. It just remains to show $\mathbf{R}_\infty\,\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})F = F$. This follows by direct computation of the integrals defining $\mathbf{R}_\infty\,\mathcal{L}(\partial_{\theta_0}, \partial_{\xi_d})F$ and the fact that when $\mathbf{E}F = 0$, we have $F_i(x, \theta_0 + \omega_i(\xi_d - \infty), \infty) = 0$. $\square$

The next Proposition summarizes what we have shown.

Proposition 1.22. *Let $F(x, \theta_0, \xi_d)$ be a function of type $\mathcal{F}$.*
(a) Then the equation $\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})\mathcal{U} = F$ has a solution bounded in (θ_0, ξ_d) if and only if $\mathbf{E}F = 0$.
(b) When $\mathbf{E}F = 0$, every C^1 solution bounded in (θ_0, ξ_d) has the form

$$\mathcal{U} = \sum_{i=1}^{3} \tau_i(x, \theta_0 + \omega_i\,\xi_d)\,r_i + \mathbf{R}_\infty F \ \textit{ with } \tau_i(x,\theta) \in C^1 \textit{ and bounded.}$$

Here $\mathbf{E}\mathcal{U} = \sum_{i=1}^3 \tau_i(x, \theta_0 + \omega_i\,\xi_d)\,r_i$ and $(I - \mathbf{E})\mathcal{U} = \mathbf{R}_\infty F$.
(c)If $\mathcal{U}$ is of type $\mathcal{F}$ then

$$\mathbf{E}\,\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})\mathcal{U} = \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})\,\mathbf{E}\mathcal{U} = 0.$$

Proof. Part (a) follows from the form of the general solution given in (1.17), and the fact that when a function F of type $\mathcal{F}$ satisfies $\mathbf{E}F = 0$, the integrals

$$\int_0^\infty F_i(x, \theta_0 + \omega_i\,(\xi_d - s), s)\,\mathrm{d}s$$

are absolutely convergent. Part (b) follows from Remark 1.19 and $\mathbf{E}\mathbf{R}_\infty F = 0$. Part (c) follows directly from Remark 1.19. $\square$

With the leading pulse profile

$$\mathcal{U}^0(x, \theta_0, \xi_d) = \sigma_1(x, \theta_0 + \omega_1\,\xi_d)\,r_1 + \sigma_2(x, \theta_0 + \omega_2\,\xi_d)\,r_2 + \sigma_3(x, \theta_0 + \omega_3\,\xi_d)\,r_3\,,$$

we can rewrite the profile system (1.22) in a form that will be useful for the error analysis as follows:

$$\begin{aligned} &a)\ \mathbf{E}\mathcal{U}^0 = \mathcal{U}^0\,,\\ &b)\ \mathbf{E}\left(\tilde{L}(\partial)\,\mathcal{U}^0 + M(\mathcal{U}^0, \partial_{\theta_0}\mathcal{U}^0) - F(0)\,\mathcal{U}^0\right) = 0\,,\\ &c)\ B(0)\,\mathcal{U}^0|_{x_d=0,\xi_d=0} = G(x', \theta_0)\,,\\ &d)\ \mathcal{U}^0 = 0 \text{ in } t < 0\,. \end{aligned} \tag{1.27}$$

These equations can also be obtained by applying the operator $\mathbf{E}$ to the equations (1.12), and have the common structure of weakly nonlinear geometric optics equations, see e.g. [20, chapters 7 and 9].

1.4 Error analysis

We end this introduction with a sketch of the error analysis used to prove Theorem 4.16, which yields Theorem 1.14 as an immediate consequence. The iteration schemes for the singular system (A.6) and the profile equations (1.27) are written side by side in (4.14), (4.15). For s large[11] and some $T_0 > 0$, the proof of Theorem 1.12 produces a sequence of iterates $U^n_\varepsilon(x,\theta_0)$, bounded in the space $E^s_{T_0}$ uniformly with respect to n and ε, and such that

$$\lim_{n\to\infty} U^n_\varepsilon = U_\varepsilon \text{ in } E^{s-1}_{T_0} \text{ uniformly with respect to } \varepsilon \in (0,\varepsilon_0]\,,$$

where U_ε is the solution of the singular system (1.5). On the other hand the construction of profiles in Proposition 3.6 yields a sequence of profile iterates $\mathcal{U}^{0,n}(x,\theta_0,\xi_d)$ bounded in $\mathcal{E}^s_{T_0}$ (see Definition 4.1) and converging in $\mathcal{E}^{s-1}_{T_0}$ to a solution $\mathcal{U}^0$ of the leading profile equations (1.27). By Proposition 4.3 this implies that the rapidly varying functions $\mathcal{U}^{0,n}_\varepsilon(x,\theta_0) := \mathcal{U}^{0,n}(x,\theta_0,\frac{x_d}{\varepsilon})$ satisfy

$$\lim_{n\to\infty} \mathcal{U}^{0,n}_\varepsilon = \mathcal{U}^0_\varepsilon \text{ in } E^{s-1}_{T_0} \text{ uniformly with respect to } \varepsilon \in (0,\varepsilon_0]\,.$$

Thus, in order to conclude $|\mathcal{U}^0_\varepsilon(x,\theta_0) - U_\varepsilon(x,\theta_0)|_{E^{s-3}_{T_0}} \le C\,\varepsilon^{\frac{1}{2M_1+5}}$ and thereby complete the proof of Theorem 1.14, it would suffice to show:

$$\text{There exists } C \text{ such that for every } n,\ |\mathcal{U}^{0,n}_\varepsilon - U^n_\varepsilon|_{E^{s-3}_{T_0}} \le C\,\varepsilon^{\frac{1}{2M_1+5}}\,. \tag{1.28}$$

The statement (1.28) is proved by induction in section 4. It is natural to try to apply the estimate of Proposition 4.17 to the difference $\mathcal{U}^{0,n+1}_\varepsilon - U^{n+1}_\varepsilon$, but the problem is that for any given n, $\mathcal{U}^{0,n+1}_\varepsilon$ does not by itself provide a very good approximate solution to the boundary problem (4.14) that defines U^{n+1}_ε. Indeed, substitution of $\mathcal{U}^{0,n+1}_\varepsilon$ into (4.14)(a) yields an error, call it $R^{n+1}_\varepsilon(x,\theta_0)$, that is $O(1)$ in $E^{s-3}_{T_0}$. Since $\mathcal{U}^{0,n+1}$ satisfies (4.15), the main contribution to $R^{n+1}_\varepsilon(x,\theta_0)$ is given by $\mathcal{R}^{n+1}(x,\theta_0,\frac{x_d}{\varepsilon})$ where

$$\mathcal{R}^{n+1} := (I-\mathbf{E})\left(\tilde{L}(\partial_x)\mathcal{U}^{0,n+1} + M(\mathcal{U}^{0,n},\partial_{\theta_0}\mathcal{U}^{0,n+1}) - F(0)\,\mathcal{U}^{0,n}\right)\,. \tag{1.29}$$

One would like to solve away the main error term in (1.29) by using Proposition 1.21 and constructing a corrector $\mathcal{U}^{1,n+1}(x,\theta_0,\xi_d)$ such that

$$\tilde{\mathcal{L}}(\partial_{\theta_0},\partial_{\xi_d})\,\mathcal{U}^{1,n+1} = -\mathcal{R}^{n+1}\,, \tag{1.30}$$

and then use a corrected approximate solution of (4.14)(a) of the form

$$\mathcal{U}^{0,n+1}_\varepsilon + \varepsilon\,\mathcal{U}^{1,n+1}_\varepsilon\,.$$

While such a corrector is given explicitly by $\mathcal{U}^{1,n+1} = \mathbf{R}_\infty(-\mathcal{R}^{n+1})$, it is not suitable for the error analysis because, although bounded, $\mathcal{U}^{1,n+1}_\varepsilon$ does not lie in any of the $E^s_{T_0}$ spaces.

To see the reason for this, note that $\mathcal{U}^{0,n}$ has the form

$$\mathcal{U}^{0,n}(x,\theta_0,\xi_d) = \sum_{i=1}^{3} \sigma^n_i(x,\theta_0+\omega_i\,\xi_d)\,r_i. \tag{1.31}$$

[11] We take $s > 1 + [M_0 + \frac{d+1}{2}]$ in Theorem 4.16.

Observe that if f is a function that decays (say like $|s|^{-2}$) as $|s| \to \infty$, the primitive $\int_\infty^\theta f(s)\,\mathrm{d}s$ itself decays to zero as $|\theta| \to \infty$ if and only if f has moment zero ($\int_{-\infty}^{\infty} f(s)\,\mathrm{d}s = 0$). Since neither $\mathcal{U}^{0,n+1}$ nor the term $M(\mathcal{U}^{0,n}, \partial_{\theta_0}\mathcal{U}^{0,n+1})$ in (1.29) has moment zero[12], the definition of $\mathbf{R}_\infty$ shows that this choice of $\mathcal{U}_\varepsilon^{1,n+1}$ generally cannot lie in any $E^s_{T_0}$ space. We first try to remedy this problem using an idea inspired by an argument in [2]. We replace $\mathcal{U}^{0,n}$ (and similarly $\mathcal{U}^{0,n+1}$) by a function $\mathcal{U}_p^{0,n}$ defined by functions $\sigma^n_{i,p}$ with vanishing first moments, where

$$\hat{\sigma}^n_{i,p}(x,m) := \chi_p(m)\,\hat{\sigma}^n_i(x,m), \quad 0 < p < 1\,,$$

and $\chi_p(m)$ is a low frequency cutoff function vanishing on a neighborhood of 0 of size $O(|p|)$ and equal to one outside a slightly larger neighborhood[13]. We show the estimate

$$|\mathcal{U}^{0,n} - \mathcal{U}_p^{0,n}|_{\mathcal{E}^{s-1}_{T_0}} \leq C\,\sqrt{p}.$$

These "moment zero approximations" (Definition 4.4) are the pulse analogues of the trigonometric polynomial approximations, which can be viewed as produced by high frequency cutoffs, used in the error analysis in the wavetrain case in [6, section 2.5]. With this change the contribution to $\mathcal{U}_\varepsilon^{1,n+1}$ from

$$-\mathbf{R}_\infty\,(I - \mathbf{E})\,\left(\tilde{L}(\partial_x)\,\mathcal{U}_p^{0,n+1} - F(0)\,\mathcal{U}_p^{0,n}\right)$$

lies in a suitable $E^r_{T_0}$ space, but there is a problem due to "self-interaction terms" of the form

$$\sigma^n_{i,p}(x,\theta)\,\partial_\theta\sigma^{n+1}_{i,p}(x,\theta)$$

coming from the M term in (1.29), which do not have moment zero. Thus, we replace these terms by $(\sigma^n_{i,p}\,\partial_\theta\sigma^{n+1}_{i,p})_p$ as in (4.28). The transversal interaction terms $\sigma^n_{i,p}\,\partial_\theta\sigma^{n+1}_{j,p}$, $i \neq j$ already yield contributions in an $E^r_{T_0}$ space (Proposition 4.10).

Using moment-zero approximations introduces errors that blow up as $p \to 0$, of course, but taking $p = \varepsilon^b$ for an appropriate $b > 0$, one can hope to control these errors using the factor ε in $\varepsilon\,\mathcal{U}^{1,n+1}$. Indeed, this works and by the process outlined above we obtain a corrector $\mathcal{U}^{1,n+1}_{p,\varepsilon}$ which, though it does not solve away R^{n+1}_ε, solves away "all but $O(\sqrt{p} + \frac{\varepsilon}{p^{M_1+2}})$" of R^{n+1}_ε in $E^{s-3}_{T_0}$ (see (4.35) for more details). Setting $p = \varepsilon^b$ and choosing the exponent b so that $\sqrt{p} = \frac{\varepsilon}{p^{M_1+2}}$ (so $b = \frac{2}{2M_1+5}$), we are able to apply the estimate of Proposition (4.17) to conclude

$$|\mathcal{U}_\varepsilon^{0,n+1} - U_\varepsilon^{n+1}|_{E^{s-3}_{T_0}} \leq C\,\varepsilon^{\frac{1}{2M_1+5}} \text{ where } M_1 = \left[\frac{d}{2} + 3\right].$$

Remark 1.23 (Uniformly stable shocks)**.** There is an analogue of Theorem 1.14 for uniformly stable shock waves perturbed by pulses. Uniform stability for the shock waves problem is an extension of Definition 1.5 and dates back to Majda [15]. The case of shocks perturbed by highly oscillatory wavetrains was studied in [6, section 3]. In that case there is a separate expansion for the oscillating shock front (a free boundary)

$$\psi_\varepsilon(x') \sim \sigma\,x_0 + \varepsilon\,\left(\chi^0(x') + \varepsilon\,\chi^1\left(x', \frac{\phi^0(x')}{\varepsilon}\right)\right)\,, \tag{1.32}$$

[12]More precisely, we refer here to the moments of profiles like $\sigma^n_i(x,\theta)$ or products of profiles that appear in these terms.

[13]The cutoff renders harmless the small divisor that appears when one writes the Fourier transform of the θ-primitive of $\sigma^n_{i,p}$ in terms of $\hat{\sigma}^n_{i,p}(x,m)$.

in addition to an expansion for the solution on each side of the front. An important difference in the pulse case is that the term $\chi^0(x')$ is absent in (1.32), and of course $\chi^1(x',\theta_0)$ is now decaying instead of periodic in θ_0. The expansions of the reflected waves on either side of the front are similar to (1.11).

The singular shock problem has the same form as in the wavetrain case (see equations (3.39) of [6] and [22]), and the profile equations again take the form of equations (3.60) in [6], except that every occurrence of $\chi^0(x')$ is replaced by 0. The solution of the large system for the leading profiles is now considerably simpler than equations (3.68) of [6], since all the interaction integrals in that equation are now absent. This reflects the fact that pulses of different families do not interact at the leading order while wavetrains do. However, it is necessary to estimate interaction integrals in the error analysis. As in the pulse problem with fixed boundaries, one can in the shock problem obtain a rate of convergence of approximate solutions to exact solutions as $\varepsilon \to 0$.

2 Exact solution of the singular problem

The goal of this section is to prove Theorem 1.12 and solve the singular system (1.5). This is achieved, as in [24, section 7], by solving the sequence of linear problems

$$
(2.1)\qquad
\begin{aligned}
&a)\ \partial_{x_d} U_\varepsilon^{n+1} + \sum_{j=0}^{d-1} \tilde{A}_j(\varepsilon U_\varepsilon^n) \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\varepsilon} \right) U_\varepsilon^{n+1} = F(\varepsilon U_\varepsilon^n)\, U_\varepsilon^n \,,\\
&b)\ B(\varepsilon U_\varepsilon^n)\, U_\varepsilon^{n+1}|_{x_d=0} = G(x',\theta_0)\,,\\
&c)\ U_\varepsilon^{n+1} = 0 \text{ in } t<0.
\end{aligned}
$$

As for the case of hyperbolic boundary value problems, that is without the singular parameter $1/\varepsilon$ in the differential operator, see e.g. [3, 4], the solvability of each linear system (2.1) relies on some a priori estimates. Our main focus here is the derivation of a priori estimates that are uniform with respect to the wavelength ε. For the reasons detailed in the introduction of [24], the appropriate functional setting in which one can derive uniform estimates is provided by the spaces E_T^s defined in (1.6). The main difficulty is to obtain L^∞ estimates uniform in ε. These cannot be obtained simply from uniform $L^2(x_d, H^2(x',\theta_0))$ estimates since the estimate of $\partial_{x_d} U_\varepsilon$ in terms of tangential derivatives provided by the system (2.1) blows up as $\varepsilon \to 0$. Much of the analysis in this section is similar to [24, sections 5 and 7], except that we use here the singular pseudodifferential calculus of Appendix A. This introduces some modifications for the regularity assumptions in the results below. In the proofs of this section we shall often refer to [24] in order to keep the exposition as short as possible.

2.1 Main estimate for the linearized singular problem

We consider a linearized problem of the form

$$
(2.2)\qquad
\begin{aligned}
&a)\ \partial_{x_d} U_\varepsilon + \sum_{j=0}^{d-1} \tilde{A}_j(\varepsilon V_\varepsilon) \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\varepsilon} \right) U_\varepsilon = f_\varepsilon \,,\\
&b)\ B(\varepsilon V_\varepsilon)\, U_\varepsilon|_{x_d=0} = g_\varepsilon \,,\\
&c)\ U_\varepsilon = 0 \text{ in } t<0,
\end{aligned}
$$

where $(V_\varepsilon)_{\varepsilon\in(0,1]}$ is a given family of functions, and $(f_\varepsilon, g_\varepsilon)$ represent some source terms. Our first main result is the analogue of [24, Theorems 5.1 and 5.2] and proves unique solvability with a

uniform L^2 energy estimate for (2.2). The main point is to keep track of the regularity assumptions on the coefficients V_ε.

Theorem 2.1. *Let $s_0 := [(d+1)/2]+1$. There exists $\delta>0$ such that, for all $K\geq 1$, there exist some constants $\gamma_0(K)\geq 1$ and $C_0(K)>0$ such that the following property holds: if the coefficients $(V_\varepsilon)_{\varepsilon\in(0,1]}$ in (2.2) satisfy*

$$|\varepsilon\, V_\varepsilon|_{L^\infty(\Omega)}\leq \delta\,, \quad |V_\varepsilon|_{C^{0,M_0}(\Omega)}+|V_\varepsilon|_{C(H^{s_0}(\mathbb{R}^d\times\mathbb{R}))}+|\varepsilon\,\partial_{x_d}V_\varepsilon|_{L^\infty(\Omega)}\leq K\,, \tag{2.3}$$

then for all $T>0$, for all source terms $f_\varepsilon\in L^2(\Omega_T)$, $g_\varepsilon\in L^2(b\Omega_T)$ vanishing for $t<0$, there exists a unique solution $U_\varepsilon\in L^2(\Omega_T)$ to (2.2) vanishing for $t<0$, and this solution satisfies

$$|\mathrm{e}^{-\gamma\, t}U_\varepsilon|_{0,0,T}+\frac{1}{\sqrt{\gamma}}\,\langle \mathrm{e}^{-\gamma\, t}U_\varepsilon|_{x_d=0}\rangle_{0,T}\leq C_0(K)\left(\frac{1}{\gamma}\,|\mathrm{e}^{-\gamma\, t}f_\varepsilon|_{0,0,T}+\frac{1}{\sqrt{\gamma}}\,\langle \mathrm{e}^{-\gamma\, t}g_\varepsilon\rangle_{0,T}\right)\,, \tag{2.4}$$

for all $\gamma\geq\gamma_0(K)$.

In Theorem 2.1, the space $C^{0,M_0}(\Omega)$ denotes the space of functions $v(x,\theta)$ such that for all $x_d\geq 0$, $v(\cdot,x_d,\cdot)$ is bounded on $\mathbb{R}^d\times\mathbb{R}$ with all derivatives up to the order M_0 bounded, and with all bounds that are uniform in x_d. The norm is defined by

$$|v|_{C^{0,M_0}(\Omega)} := \sup_{x_d\geq 0}\ \sup_{|\alpha|\leq M_0}\ \|\partial^\alpha_{x',\theta}v(\cdot,x_d,\cdot)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R})}\,.$$

For fixed x_d, the (x',θ)-regularity of a symbol enables us to use some of the symbolic calculus rules listed in Appendix A.

Proof. The first main step in the proof of Theorem 2.1 is to show a global in time a priori estimate. In other words, we consider a smooth function U_ε solution to (2.2), and wish to show the estimate (2.4) with $T=+\infty$. We begin with the following result.

Theorem 2.2 (Kreiss, Métivier [12, 19])**.** *There exists $\delta>0$ such that, if B_δ denotes the closed ball of radius δ in $\mathbb{R}^N$, there exists an $m\times m$ matrix-valued function*

$$R\in C^\infty(B_\delta\times\mathbb{R}^d\times(0,\infty)),$$

homogeneous of degree zero in (ξ',γ) and satisfying:

(a) $R(v,\xi',\gamma)=R(v,\xi',\gamma)^*$;

(b) there exist $C>0$, $c>0$ such that for all (v,ξ',γ):

$$R(v,\xi',\gamma)+C\;B^*(v)\,B(v)\geq c\,I; \tag{2.5}$$

(c) there exist finite sets of C^∞ matrices on $B_\delta\times\mathbb{R}^d\times(0,\infty)$, denoted T_l, H_l, and E_l such that

$$(i)\ \mathrm{Re}\,\left(R(v,\xi',\gamma)\,\mathcal{A}(v,\xi',\gamma)\right)=\sum_l T_l(v,\xi',\gamma)\begin{pmatrix}\gamma\,H_l(v,\xi',\gamma) & 0\\ 0 & E_l(v,\xi',\gamma)\end{pmatrix}T_l^*(v,\xi',\gamma);$$

(ii) T_l, H_l are homogeneous of degree zero in (ξ',γ), E_l is homogeneous of degree one;

(iii) $H_l(v,\xi',\gamma)=H_l^*(v,\xi',\gamma)$, $E_l(v,\xi',\gamma)=E_l^*(v,\xi',\gamma)$;

(iv) there exists $c>0$ such that

$$\sum_l T_l(v,\xi',\gamma)\,T_l^*(v,\xi',\gamma)\geq c\,I,\ H_l(v,\xi',\gamma)\geq c\,I,\ E_l(v,\xi',\gamma)\geq c\,(|\xi'|+\gamma)\,I.$$

The dimensions of H_l and E_l can vary with l.

The parameter δ is fixed according to Theorem 2.2. We then define the following singular symmetrizer for the boundary value problem (2.2):

$$\mathcal{R}_{\varepsilon,\gamma} := \mathrm{Op}^{\varepsilon,\gamma}(R(\varepsilon V_\varepsilon, \xi', \gamma)) ,$$

where singular pseudodifferential operators $\mathrm{Op}^{\varepsilon,\gamma}(a)$ associated with a symbol a are defined in Appendix A. We observe, as in [24, remark 5.2] that our symmetrizer is not self-adjoint on $L^2(\Omega)$. However, the remainder $\mathcal{R}_{\varepsilon,\gamma} - \mathcal{R}^*_{\varepsilon,\gamma}$ is $O(1/\gamma)$ as an operator on L^2, uniformly in ε.

Under the regularity assumptions (2.3) of Theorem 2.1, the results given in Appendix A and the arguments in [24, pages 164-165] give the following properties for the symmetrizer $\mathcal{R}_{\varepsilon,\gamma}$:

$$\begin{aligned}
&(a)\ |\mathcal{R}_{\varepsilon,\gamma}\, W|_{0,0} \le C(K)\, |W|_{0,0} ,\\
&(b)\ |[\partial_{x_d}, \mathcal{R}_{\varepsilon,\gamma}]\, W|_0 \le C(K)\, |W|_0 ,\\
&(c)\ \mathrm{Re}\ ((\mathcal{R}_{\varepsilon,\gamma}\, \mathcal{A}_{\varepsilon,\gamma} + \mathcal{A}^*_{\varepsilon,\gamma}\, \mathcal{R}_{\varepsilon,\gamma})\, W, W) \ge c(K)\, \gamma\, |W|^2_{0,0} ,\\
&(d)\ \mathrm{Re}\ \langle \mathcal{R}_{\varepsilon,\gamma}\, W, W\rangle + C(K)\, \langle B(\varepsilon V_\varepsilon)\, W\rangle_0^2 \ge c(K)\, \langle W\rangle_0^2 ,
\end{aligned} \tag{2.6}$$

where $\mathcal{A}_{\varepsilon,\gamma}$ denotes the operator

$$-\gamma\, \tilde{A}_0(\varepsilon V_\varepsilon) - \sum_{j=0}^{d-1} \tilde{A}_j(\varepsilon V_\varepsilon) \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\varepsilon} \right) .$$

Let us focus for instance on property (d) in (2.6). Since $\mathcal{R}^*_{\varepsilon,\gamma} = \mathcal{R}_{\varepsilon,\gamma} + O(1/\gamma)$, Gårding's inequality (Theorem A.1) shows that it is sufficient to prove that the symbol $R(\varepsilon V_\varepsilon, \xi', \gamma) + C\, B^*(\varepsilon V_\varepsilon)\, B(\varepsilon V_\varepsilon)$ is positive definite, and this property is given by (2.5). Other properties in (2.6) are obtained by similar arguments (applying Propositions A.7, A.8 or A.9), see [24, pages 164-165] for more details.

We perform the change of function $U_\varepsilon \to \mathrm{e}^{-\gamma\, t} U_\varepsilon$ in (2.2), multiply (2.2) a) by $\mathcal{R}_{\varepsilon,\gamma}$ and take the real part of the L^2 scalar product with $\mathrm{e}^{-\gamma\, t} U_\varepsilon$. The estimates (2.6) yield[14]

$$|\mathrm{e}^{-\gamma\, t} U_\varepsilon|_{0,0} + \frac{1}{\sqrt{\gamma}}\, \langle \mathrm{e}^{-\gamma\, t} U_\varepsilon|_{x_d=0}\rangle_0 \le C(K) \left(\frac{1}{\gamma}\, |\mathrm{e}^{-\gamma\, t} f_\varepsilon|_{0,0} + \frac{1}{\sqrt{\gamma}}\, \langle \mathrm{e}^{-\gamma\, t} g_\varepsilon\rangle_0 \right) ,$$

for γ sufficiently large, that is for all $\gamma \ge \gamma_0(K)$.

A similar uniform a priori estimate is valid for the dual problem (which satisfies the backward uniform Lopatinskii condition). Then the arguments of [3, 4], namely existence of a weak solution and "weak=strong" by tangential mollification, yields well-posedness of the boundary value problem (2.2). Localization in time is achieved as usual by showing a causality principle ("future does not affect the past"), which holds in our context since the constant $C(K)$ in our energy estimate is independent of γ. □

The uniform L^2 estimate (2.4) enables us to show an estimate in the space E^0 defined in (1.6). The result is similar to [24, Corollary 7.1] with a slight improvement with respect to the norm in which the source term f_ε is estimated.

Theorem 2.3. *Let $s_0 := [(d+1)/2] + 1$. There exists $\delta > 0$ such that, for all $K \ge 1$, there exist some constants $\gamma_1(K) \ge 1$ and $C_1(K) > 0$ such that the following property holds: if the coefficients $(V_\varepsilon)_{\varepsilon \in (0,1]}$ in (2.2) satisfy (2.3), then for all $T > 0$, for all source terms $f_\varepsilon \in L^2(H^1(b\Omega_T))$,*

[14] The detailed computations can be found in [24, corollary 5.2], and are the singular analogue of [3, 4].

$g_\varepsilon \in H^1(b\Omega_T)$ vanishing for $t<0$, there exists a unique solution $U_\varepsilon \in H^1(\Omega_T)$ to (2.2) vanishing for $t<0$, and this solution satisfies

$$
|\mathrm{e}^{-\gamma\, t} U_\varepsilon|_{\infty,0,T} + |\mathrm{e}^{-\gamma\, t} U_\varepsilon|_{0,1,T} + \frac{1}{\sqrt{\gamma}} \, \langle \mathrm{e}^{-\gamma\, t} U_\varepsilon|_{x_d=0} \rangle_{1,T} \le C_1(K) \left(\frac{1}{\gamma} \, |\mathrm{e}^{-\gamma\, t} f_\varepsilon|_{0,1,T} + \frac{1}{\sqrt{\gamma}} \, \langle \mathrm{e}^{-\gamma\, t} g_\varepsilon \rangle_{1,T} \right) , \tag{2.7}
$$

for all $\gamma \ge \gamma_1(K)$.

Proof. The regularity of the solution U_ε can be obtained by using the same arguments as in [4, chapter 7], that is by commuting the system (2.2) with a mollified version of the Fourier multiplier of symbol $(\gamma^2 + |\xi'|^2 + k^2)^{1/2}$. The argument shows that the $\partial_{x'}$ and ∂_θ derivatives of U_ε are in L^2, and (2.2) then shows that the ∂_{x_d} derivative of U_ε also belongs to L^2. We thus only show the estimate (2.7).

1. L^2 estimate of tangential derivatives. Commuting (2.2) with a tangential derivative $\partial_{tan} \in \{\partial_{x_0}, \dots, \partial_{x_{d-1}}, \partial_{\theta_0}\}$, we need to control the commutators

$$
\sum_{j=0}^{d-1} [\tilde{A}_j(\varepsilon\, V_\varepsilon), \partial_{tan}] \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\varepsilon} \right) U_\varepsilon = \sum_{j=0}^{d-1} \left(\mathrm{d}\tilde{A}_j(\varepsilon\, V_\varepsilon) \cdot \partial_{tan} V_\varepsilon \right) \left(\varepsilon\, \partial_{x_j} + \beta_j\, \partial_{\theta_0} \right) U_\varepsilon \, .
$$

When multiplied by $\mathrm{e}^{-\gamma t}$, this source term is bounded in $L^2(\Omega_T)$ by a constant times $|\mathrm{e}^{-\gamma\, t} U_\varepsilon|_{0,1,T}$ and can therefore be absorbed from right to left by choosing γ large. At this stage, we have

$$
|\mathrm{e}^{-\gamma\, t} U_\varepsilon|_{0,1,T} + \frac{1}{\sqrt{\gamma}} \, \langle \mathrm{e}^{-\gamma\, t} U_\varepsilon|_{x_d=0} \rangle_{1,T} \le C_1(K) \left(\frac{1}{\gamma} \, |\mathrm{e}^{-\gamma\, t} f_\varepsilon|_{0,1,T} + \frac{1}{\sqrt{\gamma}} \, \langle \mathrm{e}^{-\gamma\, t} g_\varepsilon \rangle_{1,T} \right) , \tag{2.8}
$$

for all γ large enough.

2. $L^\infty(L^2)$ estimate, part 1. We extend f_ε and g_ε beyond time T, which does not affect the solution U_ε up to time T. Doing so, we just need to prove the $L^\infty(L^2)$ estimate (2.7) for $T = +\infty$. We consider a cut-off function χ^e in the extended singular calculus, that is a smooth function satisfying the conditions (A.3) given in Appendix A. The $L^\infty(L^2)$ estimate is first proved on $(1 - \chi^e_s(D))\, (\mathrm{e}^{-\gamma\, t} U_\varepsilon)$, where we let from now on $\chi^e_s(D)$ denote the Fourier multiplier whose symbol is

$$
\chi^e \left(\xi', \frac{k\, \beta}{\varepsilon}, \gamma \right) .
$$

Since $|k\, \beta|/\varepsilon$ is dominated by $(\gamma^2 + |\xi'|^2)^{1/2}$ on the support of $1 - \chi^e$, the same arguments as in [24, page 173] yield

$$
\begin{aligned}
|(1 - \chi^e_s(D))\, (\mathrm{e}^{-\gamma\, t} U_\varepsilon)|_{\infty,0} &\le C(K) \left(|\mathrm{e}^{-\gamma\, t} f_\varepsilon|_{0,0} + |\mathrm{e}^{-\gamma\, t} U_\varepsilon|_{0,1} \right) \\
&\le C(K) \left(\frac{1}{\gamma} \, |\mathrm{e}^{-\gamma\, t} f_\varepsilon|_{0,1} + \frac{1}{\sqrt{\gamma}} \, \langle \mathrm{e}^{-\gamma\, t} g_\varepsilon \rangle_{1} \right) .
\end{aligned} \tag{2.9}
$$

3. $L^\infty(L^2)$ estimate, part 2. It remains to estimate $|\chi^e_s(D)\, (\mathrm{e}^{-\gamma\, t} U_\varepsilon)|_{\infty,0}$, which uses the fact that β is a hyperbolic frequency. More precisely, we can fix some parameters $\delta > 0$ and $\delta_2 > 0$ such that for all v in the ball of radius δ and for all (z, η) that are δ_2-close to β, there holds

$$
Q(v,z,\eta)^{-1} \, \mathcal{A}(v,z,\eta)\, Q(v,z,\eta) = \mathrm{diag}\, \left(\lambda_1(v,z,\eta), \dots, \lambda_N(v,z,\eta) \right) ,
$$

for a suitable invertible matrix Q, and the λ_j's satisfy

$$
\mathrm{Re}\ \lambda_j(v,z,\eta) \begin{cases} \le -c\, \gamma \, , & \text{if } j = 1, \dots, p, \\ \ge c\, \gamma \, , & \text{if } j = p+1, \dots, N. \end{cases}
$$

Moreover, Assumption 1.6 shows that the (square) matrix whose column vectors are

$$B(v)\,Q_1(v,z,\eta),\dots,B(v)\,Q_p(v,z,\eta)\,,$$

is invertible (here the Q_j's denote the columns of Q).

With the above notation, we can follow the proof of [24, Proposition 7.3], and write $\chi_s^e(D)\,(\mathrm{e}^{-\gamma\,t}U_\varepsilon)$ under the form

$$\chi_s^e(D)\,(\mathrm{e}^{-\gamma\,t}U_\varepsilon)=r_0\,\mathcal{W}\,,$$

where, here and from now on, r_0 denotes a bounded operator on $L^2(\Omega)$ whose operator norm is independent of ε,γ, and where each component $\mathcal{W}_j$ of $\mathcal{W}$ satisfies a transport equation

$$\partial_{x_d}\mathcal{W}_j-\lambda_j(\varepsilon\,V_\varepsilon,D_s)\,\mathcal{W}_j=r_0(\mathrm{e}^{-\gamma\,t}f_\varepsilon)+r_0(\mathrm{e}^{-\gamma\,t}U_\varepsilon)\,. \tag{2.10}$$

In (2.10), $\lambda_j(\varepsilon\,V_\varepsilon,D_s)$ denotes the singular pseudodifferential operator of symbol $\lambda_j(\varepsilon\,V_\varepsilon,z,\eta)$ (as described in Appendix A).

In the outgoing case ($j=p+1,\dots,N$), we multiply (2.10) by $\overline{\mathcal{W}_j}$, integrate from x_d to $+\infty$ and apply Gårding's inequality (Theorem A.1), obtaining

$$\begin{aligned}\langle\mathcal{W}_j(x_d)\rangle_0^2+\gamma\int_{x_d}^{+\infty}&\langle\mathcal{W}_j(y)\rangle_0^2\,\mathrm{d}y\\&\le C\int_{x_d}^{+\infty}\langle\mathcal{W}_j(y)\rangle_0\,\langle\mathrm{e}^{-\gamma\,t}f_\varepsilon(y)\rangle_0\,\mathrm{d}y+C\int_{x_d}^{+\infty}\langle\mathcal{W}_j(y)\rangle_0\,\langle\mathrm{e}^{-\gamma\,t}U_\varepsilon(y)\rangle_0\,\mathrm{d}y\,.\end{aligned}$$

The contribution of $\mathcal{W}_j$ on the right-hand side can be absorbed on the left by using Young's inequality, and Theorem 2.1 enables us to control the L^2 norm of U_ε. We thus get

$$\sup_{j=p+1,\dots,N}|\mathcal{W}_j|_{\infty,0}^2\le C(K)\left(\frac{1}{\gamma}\,|\mathrm{e}^{-\gamma\,t}f_\varepsilon|_{0,0}^2+\frac{1}{\gamma^2}\,\langle\mathrm{e}^{-\gamma\,t}g_\varepsilon\rangle_0^2\right)\,. \tag{2.11}$$

The estimates in the incoming case are similar, except that we integrate from 0 to x_d. We thus get

$$\sup_{j=1,\dots,p}|\mathcal{W}_j|_{\infty,0}^2\le\sup_{j=1,\dots,p}\langle\mathcal{W}_j|_{x_d=0}\rangle_0^2+C(K)\,\left(\frac{1}{\gamma}\,|\mathrm{e}^{-\gamma\,t}f_\varepsilon|_{0,0}^2+\frac{1}{\gamma^2}\,\langle\mathrm{e}^{-\gamma\,t}g_\varepsilon\rangle_0^2\right)\,.$$

Using the same arguments as in [24, page 178], we can use the uniform Lopatinskii condition and write

$$\begin{pmatrix}\mathcal{W}_1|_{x_d=0}\\ \vdots\\ \mathcal{W}_p|_{x_d=0}\end{pmatrix}=r_0\,\begin{pmatrix}\mathcal{W}_{p+1}|_{x_d=0}\\ \vdots\\ \mathcal{W}_N|_{x_d=0}\end{pmatrix}+r_0\,(\mathrm{e}^{-\gamma\,t}g_\varepsilon)+\frac{1}{\gamma}\,r_0\,(\mathrm{e}^{-\gamma\,t}U_\varepsilon|_{x_d=0})\,,$$

from which we derive the estimate

$$\sup_{j=1,\dots,p}\langle\mathcal{W}_j|_{x_d=0}\rangle_0^2\le C(K)\sup_{j=p+1,\dots,N}|\mathcal{W}_j|_{\infty,0}^2+C(K)\,\langle\mathrm{e}^{-\gamma\,t}g_\varepsilon\rangle_0^2+\frac{C(K)}{\gamma^3}\,|\mathrm{e}^{-\gamma\,t}f_\varepsilon|_{0,0}^2\,.$$

We combine the latter inequality with (2.11), and recall $\chi_s^e(D)\,(\mathrm{e}^{-\gamma\,t}U_\varepsilon)=r_0\,\mathcal{W}$, so we get

$$|\chi_s^e(D)\,(\mathrm{e}^{-\gamma\,t}U_\varepsilon)|_{\infty,0}\le C(K)\,\left(\frac{1}{\sqrt{\gamma}}\,|\mathrm{e}^{-\gamma\,t}f_\varepsilon|_{0,0}^2+\langle\mathrm{e}^{-\gamma\,t}g_\varepsilon\rangle_0^2\right)\,.$$

Adding with (2.9) and (2.8), we complete the proof of Theorem 2.3. □

2.2 Construction of the exact solution

We use the iteration scheme (2.1) to solve the nonlinear system (1.5). As usual, the convergence of the iteration scheme follows from the combination of two arguments: a uniform boundedness in a "high norm" (here in the space E^s given in (1.6)), and a contraction property in a "low norm" (here in E^0). The estimate of a solution in E^0 will be provided by Theorem 2.3 above, and we indicate below how we obtain the estimate of a solution in E^s, $s \in \mathbb{N}$.

Proposition 2.4. *Let $s_0 := [(d+1)/2] + 1$ and let $k \in \mathbb{N}$. There exists $\delta > 0$ such that, for all $K \geq 1$, there exist some constants $\gamma_k(K) \geq 1$ and $C_k(K) > 0$ such that the following property holds: if the coefficients $(V_\varepsilon)_{\varepsilon \in (0,1]}$ in (2.2) satisfy (2.3) and belong to $L^2(H^{k+1}(b\Omega_T)) \cap L^\infty(H^k(b\Omega_T))$, then for all $T > 0$, for all source terms $f_\varepsilon \in L^2(H^{k+1}(b\Omega_T))$, $g_\varepsilon \in H^{k+1}(b\Omega_T)$ vanishing for $t < 0$, there exists a unique solution $U_\varepsilon \in L^2(H^{k+1}(b\Omega_T)) \cap L^\infty(H^k(b\Omega_T))$ to (2.2) vanishing for $t < 0$, and this solution satisfies*

$$(2.12) \quad |\mathrm{e}^{-\gamma t} U_\varepsilon|_{\infty,k,T} + |\mathrm{e}^{-\gamma t} U_\varepsilon|_{0,k+1,T} + \frac{1}{\sqrt{\gamma}} \langle \mathrm{e}^{-\gamma t} U_\varepsilon|_{x_d=0} \rangle_{k+1,T} \leq C_k(K) \left(\frac{1}{\gamma} |\mathrm{e}^{-\gamma t} f_\varepsilon|_{0,k+1,T} \right.$$
$$\left. + \frac{1}{\sqrt{\gamma}} \langle \mathrm{e}^{-\gamma t} g_\varepsilon \rangle_{k+1,T} + |U_\varepsilon|_{L^\infty(W^{1,\infty}(b\Omega_T))} \left(\frac{|\mathrm{e}^{-\gamma t} V_\varepsilon|_{0,k+1,T}}{\gamma} + \frac{|\mathrm{e}^{-\gamma t} V_\varepsilon|_{x_d=0}|_{k+1,T}}{\sqrt{\gamma}} \right) \right),$$

for all $\gamma \geq \gamma_k(K)$.

Proof. The proof is like that of [24, Theorem 7.2]. One commutes (2.2) with a tangential derivative ∂^α of order $1 \leq |\alpha| \leq k$, and applies (tangential) Gagliardo-Nirenberg inequalities. When the additional fast variable θ lies in the torus $\mathbb{R}/\mathbb{Z}$, these inequalities are given in [24, Lemma 7.3], and we claim that the exact same inequalities are valid when the fast variable lies in $\mathbb{R}$.

When commuting (2.2) with a tangential derivative ∂^α, one applies Theorem 2.3 and needs to control the commutator

$$\left[\tilde{A}_j(\varepsilon V_\varepsilon) \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\varepsilon} \right) ; \partial^\alpha \right] U_\varepsilon ,$$

in the norm $|\mathrm{e}^{-\gamma t} \cdot |_{0,1,T}$. The ε factor in front of V_ε cancels the singular $1/\varepsilon$ factor and we obtain the estimate

$$\left| \mathrm{e}^{-\gamma t} \left[\tilde{A}_j(\varepsilon V_\varepsilon) \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\varepsilon} \right) ; \partial^\alpha \right] U_\varepsilon \right|_{0,1,T} \leq C(K) \, |\mathrm{e}^{-\gamma t} U_\varepsilon|_{0,k+1,T}$$
$$+ C(K) \, |U_\varepsilon|_{L^\infty(W^{1,\infty}(b\Omega_T))} \, |\mathrm{e}^{-\gamma t} V_\varepsilon|_{0,k+1,T} .$$

The $|\mathrm{e}^{-\gamma t} U_\varepsilon|_{0,k+1,T}$ term on the right hand-side is absorbed by choosing γ large enough, and we are left with (2.12). (Estimates on the boundary are similar.) $\square$

We can then deduce the main estimate in the space E^k_T defined in (1.6) (the proof is the same as that of [24, Corollary 7.2] and is based on the choice $T = 1/\gamma$ in Proposition 2.4).

Corollary 2.5. *Let $k \geq M_0 + [\frac{d+1}{2}]$ and $K_1, K_2 \geq 1$. Then there exist a constant $C(K_1,K_2) > 0$, a parameter $\varepsilon_0(K_1,K_2) \in (0,1]$ and a time $\mathcal{T}(K_1,K_2) > 0$ satisfying the following property: if $T \leq \mathcal{T}(K_1,K_2)$, if the coefficients $(V_\varepsilon)_{\varepsilon \in (0,1]}$ in (2.2) belong to E^k_T and satisfy*

$$(2.13) \qquad |V_\varepsilon|_{\infty,k,T} + |V_\varepsilon|_{x_d=0}|_{k+1,T} \leq K_1 , \quad |\varepsilon \, \partial_{x_d} V_\varepsilon|_{L^\infty(\Omega_T)} \leq K_2 ,$$

and if $\varepsilon \leq \varepsilon_0(K_1, K_2)$, then for all source terms $f_\varepsilon \in L^2(H^{k+1}(b\Omega_T))$, $g_\varepsilon \in H^{k+1}(b\Omega_T)$ vanishing for $t < 0$, there exists a unique solution $U_\varepsilon \in E^k_T$ to (2.2) vanishing for $t < 0$, and this solution satisfies

$$(2.14)\quad |U_\varepsilon|_{\infty,k,T} + |U_\varepsilon|_{0,k+1,T} + \sqrt{T}\,\langle U_\varepsilon|_{x_d=0}\rangle_{k+1,T} \leq C_k(K_1, K_2)\left(T\,|f_\varepsilon|_{0,k+1,T} + \sqrt{T}\,\langle g_\varepsilon\rangle_{k+1,T}\right).$$

The parameter ε_0 in Corollary 2.5 is chosen so that (2.13) implies $|\varepsilon\, V_\varepsilon|_{L^\infty(\Omega_T)} \leq \delta$ where δ is as in Proposition 2.4.

We are now in a position to prove our main existence result for the singular system (1.5). The norm in the space E^k_T is defined by

$$|v|_{E^k_T} := |v|_{\infty,k,T} + |v|_{0,k+1,T}\,.$$

Theorem 2.6. *Let $K > 0$ and let $k \geq M_0 + [\frac{d+1}{2}]$. Then there exists a constant $K' > 0$, a parameter $\varepsilon_0(K) \in (0, 1]$ and a time $\mathcal{T}(K) > 0$ satisfying the following property: the iteration (2.1) with $U^0_\varepsilon \equiv 0$ is well-defined for $0 < T \leq \mathcal{T}(K)$ and satisfies*

$$\forall\, n \in \mathbb{N}\,, \quad \forall \varepsilon \leq \varepsilon_0(K)\,, \quad |U^n_\varepsilon|_{E^k_T} + |U^n_\varepsilon|_{x_d=0}|_{k+1,T} \leq K\,, \quad |\varepsilon\,\partial_{x_d} U^n_\varepsilon|_{L^\infty(\Omega_T)} \leq K'\,.$$

Moreover, the sequence (U^n_ε) converges towards a function U_ε in E^{k-1}_T, uniformly with respect to $\varepsilon \in (0, \varepsilon_0(K)]$. The limit U_ε belongs to E^k_T and is a solution to (1.5).

Proof. The constant K' is chosen such that, if $|U^n_\varepsilon|_{E^k_T} \leq K$, and if furthermore $|V_\varepsilon|_{E^k_T} \leq K$, then one has

$$(2.15)\qquad \left|\varepsilon\, F(\varepsilon\, U^n_\varepsilon) U^n_\varepsilon - \sum_{j=0}^{d-1} \tilde{A}_j(\varepsilon\, U^n_\varepsilon)(\varepsilon\,\partial_{x_j} + \beta_j\,\partial_{\theta_0}) V_\varepsilon\right|_{L^\infty(\Omega_T)} \leq K'\,,$$

independently of $\varepsilon \in (0, 1]$. Then the parameter ε_0 is chosen as $\varepsilon_0(K, K')$ given by Corollary 2.5. The time $\mathcal{T}(K, K') > 0$ is chosen accordingly. Assuming that the induction assumption

$$\forall\, j \leq n\,, \quad \forall\, \varepsilon \leq \varepsilon_0(K)\,, \quad |U^j_\varepsilon|_{E^k_T} + \langle U^j_\varepsilon|_{x_d=0}\rangle_{k+1,T} \leq K\,, \quad |\varepsilon\,\partial_{x_d} U^j_\varepsilon|_{L^\infty(\Omega_T)} \leq K'\,,$$

holds (this is trivially true for $n = 0$), we can apply the estimate (2.14) of Corollary 2.5 to the system (2.1) and obtain

$$\begin{aligned} |U^{n+1}_\varepsilon|_{E^k_T} + \sqrt{T}\,\langle U^{n+1}_\varepsilon|_{x_d=0}\rangle_{k+1,T} &\leq C_k(K, K')\left(T\,|F(\varepsilon U^n_\varepsilon) U^n_\varepsilon|_{0,k+1,T} + \sqrt{T}\,\langle G\rangle_{k+1,T}\right) \\ &\leq C_k(K, K')\left(T\, C(K) + \sqrt{T}\,\langle G\rangle_{k+1,T}\right). \end{aligned}$$

Since $\langle G\rangle_{k+1,T}$ tends to zero as T tends to zero, we can choose the time T small enough so that the induction assumption implies

$$\forall\, \varepsilon \leq \varepsilon_0(K)\,, \quad |U^{n+1}_\varepsilon|_{E^k_T} + \langle U^{n+1}_\varepsilon|_{x_d=0}\rangle_{k+1,T} \leq K\,.$$

Our choice of K' in (2.15) implies that the induction assumption propagates from the rank n to the rank $n + 1$ because $|\varepsilon\,\partial_{x_d} U^{n+1}_\varepsilon|_{L^\infty(\Omega_T)} \leq K'$.

The convergence in E^{k-1}_T is obtained by showing a contraction estimate in E^0_T, which is obtained by applying Theorem 2.3. We refer to [24, page 184] for the details. The limit U_ε of the iteration scheme (2.1) is a solution to (1.5), which yields the regularity $U_\varepsilon \in E^k_T$ (see [3, chapter 9] for similar arguments). $\square$

3 Construction of the leading pulse profiles

Observe that we can solve the system (1.22) by solving instead

$$X_{\phi_i}\sigma_i + c_i^i\,\sigma_i\,\partial_\theta\sigma_i - e_i^i\,\sigma_i = 0,\ i=1,2,3 \tag{3.1}$$
$$(\sigma_i(x',0,\theta), i=1,3) = \mathcal{B}\left(G(x',\theta), \sigma_2(x',0,\theta)\right),$$
$$\sigma_i = 0 \text{ in } t<0.$$

where all occurrences of θ_i or θ_0 are now replaced by θ. To solve (3.1) we use the iteration scheme

$$(a)\ X_{\phi_i}\sigma_i^{n+1} + c_i^i\,\sigma_i^n\partial_\theta\sigma_i^{n+1} = e_i^i\,\sigma_i^n,\ i=1,2,3 \tag{3.2}$$
$$(b)\ (\sigma_i^{n+1}(x',0,\theta), i=1,3) = \mathcal{B}\left(G(x',\theta), \sigma_2^{n+1}(x',0,\theta)\right),$$
$$(c)\ \sigma_i^{n+1} = 0 \text{ in } t<0.$$

We will prove estimates for (3.2) in a class of Sobolev spaces weighted in θ. These weights are introduced in order to get an explicit decay rate in θ at infinity.

Definition 3.1. *For $s\in\mathbb{N}$ and $\gamma\geq 1$ define the spaces*

$$\Gamma^s := \left\{a(x,\theta)\in L^2(\mathbb{R}^{d+1}_+\times\mathbb{R}) : (\theta,\partial_x,\partial_\theta)^\beta a\in L^2 \text{ for } |\beta|\leq s, \text{ and } a=0 \text{ in } t<0\right\},$$
$$\text{and } \Gamma^s_\gamma := e^{\gamma t}\,\Gamma^s,$$

with respective norms

$$|a|_s := \sum_{|\beta|=|(\beta_1,\beta_2,\beta_3)|\leq s} |\theta^{\beta_1}\,\partial_x^{\beta_2}\,\partial_\theta^{\beta_3}\,a|_{L^2(x,\theta)} \quad \text{and } |a|_{s,\gamma} := |e^{-\gamma t}a|_s. \tag{3.3}$$

We will let H^s and H^s_γ denote the usual Sobolev spaces with norms defined just as in (3.3) *but without the θ weights. These spaces and those below have the obvious meanings when a is vector-valued.*

Remark 3.2. We have

$$|a|_{s,\gamma} \sim \sum_{|\beta|\leq s}\gamma^{s-|\beta|}\,|\mathrm{e}^{-\gamma t}\,\theta^{\beta_1}\,\partial_x^{\beta_2}\,\partial_\theta^{\beta_3}a|_{L^2(x,\theta)} \sim \sum_{|\beta|\leq s}|\mathrm{e}^{-\gamma t}\,\theta^{\beta_1}\,\partial_x^{\beta_2}\,\partial_\theta^{\beta_3}a|_{L^2(x,\theta)},$$

where "$\sim$" denotes an equivalence of norms with constants independent of $\gamma\geq 1$. The second equivalence follows from

$$(\partial_t+\gamma)\,(\mathrm{e}^{-\gamma t}\,a) = \mathrm{e}^{-\gamma t}\,\partial_t a.$$

The next proposition is helpful for estimating the commutators that arise when deriving Γ^s estimates of solutions to the linearization of the profile system (1.22). Define

$$\Lambda^s := \Big\{a\in L^2(\mathbb{R}^{d+1}_+\times\mathbb{R}) :\quad \theta^{\beta_1}\,a\in L^2 \text{ for } |\beta_1|\leq s,\quad \partial_x^{\beta_2}\,a\in L^2 \text{ for } |\beta_2|\leq s,$$
$$\partial_\theta^{\beta_3}\,a\in L^2 \text{ for } |\beta_3|\leq s,\quad a=0 \text{ in } t<0\Big\},$$

with

$$|a|_{\Lambda^s} := \sum_{|\beta_1|\leq s}|\theta^{\beta_1}\,a|_{L^2} + \sum_{|\beta_2|\leq s}|\partial_x^{\beta_2}\,a|_{L^2} + \sum_{|\beta_3|\leq s}|\partial_\theta^{\beta_3}\,a|_{L^2},$$

and let define $\Lambda^s_\gamma := \mathrm{e}^{\gamma t}\,\Lambda^s$ with the norm $|a|_{\Lambda^s_\gamma} := |\mathrm{e}^{-\gamma t}a|_{\Lambda^s}$ accordingly.

Proposition 3.3. *The spaces Γ^s and Λ^s are equal, and the norms $|a|_s$ and $|a|_{\Lambda^s}$ are equivalent: there exists a constant C_s such that*

$$|a|_{\Lambda^s} \leq |a|_s \leq C_s\, |a|_{\Lambda^s}\,. \tag{3.4}$$

Proof. Clearly, $|a|_{\Lambda^s} \leq |a|_s$. The remaining inequality is proved by induction on s. The case $s = 0$ is clear. The square $|a|_s^2$ is a sum of terms

$$\int (\theta^{\beta_1}\, \partial_x^{\beta_2}\, \partial_\theta^{\beta_3}\, a)\, (\theta^{\beta_1}\, \partial_x^{\beta_2}\, \partial_\theta^{\beta_3} a)\, \mathrm{d}x\, \mathrm{d}\theta\,, \tag{3.5}$$

where $|\beta| = |\beta_1| + |\beta_2| + |\beta_3| \leq s$. The terms with $|\beta| < s$ are dominated by $C|a|_{\Lambda^s}^2$ by the induction assumption.

Consider now a term like (3.5) with $|\beta| = s > 0$. Either $2\,|\beta_1| \geq s$ or $2\,|\beta_2, \beta_3| \geq s$. Suppose $2\,|\beta_2, \beta_3| \geq s$. Perform integrations by parts to obtain terms of the form

$$C \int \partial_{(x,\theta)}^{\alpha_1}\, a \cdot (\theta, \partial_x, \partial_\theta)^{\alpha_2}\, a\ \mathrm{d}x\, \mathrm{d}\theta \leq C_\delta\, |\partial_{(x,\theta)}^{\alpha_1}\, a|_{L^2}^2 + \delta\, |(\theta, \partial_x, \partial_\theta)^{\alpha_2}\, a|_{L^2}^2, \quad |\alpha_i| = s,\ i = 1, 2\,, \tag{3.6}$$

and other terms (where powers of θ are differentiated) that can be estimated using the induction assumption. The second term on the right in (3.6) can be absorbed by $|a|_s^2$. Integrations by parts show that the first term on the right is dominated by the sum of a multiple of $|a|_{\Lambda^s}^2$ and a term that can be absorbed by $|a|_s^2$.

The remaining case $2|\beta_1| \geq s$ is handled similarly. $\square$

Remark 3.4. 1) The spaces Γ^s, Γ^s_γ, Λ^s, and Λ^s_γ all have obvious analogues when $L^2(\mathbb{R}^{d+1}_+ \times \mathbb{R})$ is replaced by $L^2(\Omega_T)$ in the definitions, where we recall the notation

$$\Omega_T = \big\{(x, \theta) \in \mathbb{R}^{d+1}_+ \times \mathbb{R} : t < T\big\}.$$

The corresponding norms are denoted by adding the subscript T: $|a|_{s,T}$, $|a|_{s,\gamma,T}$, $|a|_{\Lambda^s_T}$, etc. The equivalence (3.4) continues to hold for the norms restricted to Ω_T, as can be seen by a standard argument using Seeley extensions [4].

2) The analogous norms of functions of $b(x', \theta)$ defined on $\mathbb{R}^d \times \mathbb{R}$ or on $b\Omega_T$ are denoted with brackets: $\langle b\rangle_{s,T}$, $\langle b\rangle_{s,\gamma,T}$, etc. We denote the corresponding spaces by $b\Gamma^s_T$, $b\Lambda^s_T$, etc.

3) By Sobolev embedding it follows that if the functions σ_j appearing in (1.19) lie in Γ^s_T for $s > \frac{d+2}{2} + 3$, then $\mathcal{F}$ as in (1.15) is of type $\mathcal{F}$. Indeed, we then have, for example, $\theta_j^2\, \partial_{\theta_j} \sigma_j \in H^t_T$ for an index $t > \frac{d+2}{2}$.

4) More generally, if the functions f^i_k, $g^i_{l,m}$, $h^i_{l,m}$ appearing in (1.24) lie in Γ^s_T for $s > \frac{d+2}{2} + 2$, then F as in (1.23) is of type $\mathcal{F}$.

We set $|a|_\infty := |a|_{L^\infty(\Omega_T)}$ when the domain Ω_T makes no possible confusion, and we define

$$W^{1,\infty}_T := \Big\{a(x, \theta)\ :\ |a|_{1,\infty} := \sum_{|\alpha| \leq 1} |\partial^\alpha_{x,\theta} a|_\infty < \infty\Big\}\,.$$

Estimates for the coupled systems. We can now state the main existence result for solutions

$$\mathcal{V}^{0,n+1}(x, \theta) = (\sigma_1^{n+1}, \sigma_2^{n+1}, \sigma_3^{n+1}) \tag{3.7}$$

to the sequence of linear systems (3.2).

Proposition 3.5. *Let $T > 0$, $m > \frac{d+2}{2} + 1$ and suppose that $G \in b\Gamma_T^m$ and $\mathcal{V}^{0,n} \in \Gamma_T^m$ both vanish in $t \leq 0$. Then the system (3.2) has a unique solution $\mathcal{V}^{0,n+1} \in \Gamma_T^m$ vanishing in $t \leq 0$ with $\sigma_2^{n+1} \equiv 0$. Moreover, there exist increasing functions, $\gamma_0(K)$ and $C(K)$ of $K := |\mathcal{V}^{0,n}|_{m,T}$ such that for $\gamma \geq \gamma_0(K)$ we have*

$$|\mathcal{V}^{0,n+1}|_{m,\gamma,T} + \frac{\langle \mathcal{V}^{0,n+1} \rangle_{m,\gamma,T}}{\sqrt{\gamma}} \leq C(K) \left(\frac{\langle G \rangle_{m,\gamma,T}}{\sqrt{\gamma}} + \frac{|\mathcal{V}^{0,n}|_{m,\gamma,T}}{\gamma} \right). \tag{3.8}$$

Proof. **1. L^2 estimate.** Anticipating the extra forcing terms that arise in the higher derivative estimates, we first prove an L^2 a priori estimate in the case where a forcing term $f_i(x,\theta)$ vanishing in $t \leq 0$ is added to the right side of each interior equation in (3.2). Setting $F(x,\theta) := (f_1, f_2, f_3)$ we claim

$$|\mathcal{V}^{0,n+1}|_{0,\gamma,T} + \frac{\langle \mathcal{V}^{0,n+1} \rangle_{0,\gamma,T}}{\sqrt{\gamma}} \leq C(K') \left(\frac{|F|_{0,\gamma,T}}{\gamma} + \frac{\langle G \rangle_{0,\gamma,T}}{\sqrt{\gamma}} + \frac{|\mathcal{V}^{0,n}|_{0,\gamma,T}}{\gamma} \right), \tag{3.9}$$

where $K' := |\mathcal{V}^{0,n}|_{1,\infty}$. The latter estimate is obtained by considering the weighted function $\mathrm{e}^{-\gamma t}\, \mathcal{V}^{0,n+1}$, and by performing straightforward energy estimates on (3.2)(a). The traces of σ_i^{n+1}, $i = 1, 3$, are directly estimated by using (3.2)(b).

2. Higher order estimates. We use again the system (3.2) in its original form. Using the equivalence of norms established in Proposition 3.3, we first apply the L^2 estimate to the problems satisfied by $\theta^k\, \sigma_i$, where $k \leq m$. The forcing term in this case is

$$-c_i^i\, [\sigma_i^n\, \partial_\theta, \theta^k]\, \sigma_i^{n+1} = -c_i^i\, k\, \sigma_i^n\, \theta^{k-1}\, \sigma_i^{n+1}.$$

Clearly we may assume $|\theta| \geq 1$. Applying (3.9) we can absorb the terms on the right involving σ_i^{n+1} to obtain

$$|\theta^k\, \mathcal{V}^{0,n+1}|_{0,\gamma,T} + \frac{\langle \theta^k\, \mathcal{V}^{0,n+1} \rangle_{0,\gamma,T}}{\sqrt{\gamma}} \leq C(K') \left(\frac{\langle G \rangle_{m,\gamma,T}}{\sqrt{\gamma}} + \frac{|\mathcal{V}^{0,n}|_{m,\gamma,T}}{\gamma} \right) \text{ for } \gamma \geq \gamma_0(K'). \tag{3.10}$$

Next we estimate $\partial_{x'}^\alpha \sigma_i^{n+1}$ for $|\alpha| \leq m$. The forcing term in the problem satisfied by $\partial_{x'}^\alpha \sigma_i^{n+1}$ is now

$$-c_i^i\, [\sigma_i^n\, \partial_\theta, \partial_{x'}^\alpha]\, \sigma_i^{n+1}.$$

The commutator is a finite linear combination of terms of the form

$$(\partial_{x'}^{\alpha_1} \sigma_i^n)\, (\partial_{x'}^{\alpha_2} \partial_\theta \sigma_i^{n+1}), \;\; |\alpha_1| + |\alpha_2| = |\alpha|, \;\; |\alpha_1| \geq 1. \tag{3.11}$$

We estimate these terms using the following two observations:

A. Suppose $m_1 + m_2 > \frac{d+2}{2}$, $m_i \geq 0$. Then the product $(a(x,\theta), b(x,\theta)) \to a \cdot b$ is continuous from $H_T^{m_1} \times H_T^{m_2} \to H_T^0$.

B. Suppose $|\alpha_1| + |\alpha_2| \leq |\alpha| \leq m$. Then

$$|\mathrm{e}^{-\gamma t}\, \partial_{x,\theta}^{\alpha_1} u(x,\theta)|_{|\alpha_2|,T} \leq |u|_{m,\gamma,T}.$$

By **A** (with $m_1 = m - |\alpha_1|$, $m_2 = |\alpha_1| - 1$) and **B** we have

$$|(3.11)|_{0,\gamma,T} \leq C\, |\sigma_i^n|_{m,T}\, |\sigma_i^{n+1}|_{m,\gamma,T} \leq C\, K\, |\sigma_i^{n+1}|_{m,\gamma,T}.$$

Applying (3.9) and absorbing terms from the right, we obtain an estimate like (3.10) for $\partial_{x'}^{\alpha}\mathcal{V}^{0,n+1}$ with $C(K')$ replaced by $C(K)$.

The θ derivatives $\partial_\theta^k\sigma_i$, $k\leq m$, are estimated similarly. Derivatives involving ∂_{x_d} are estimated in the customary way using the tangential estimates and the fact that $x_d=0$ is noncharacteristic for X_{ϕ_i}.

3. Existence and uniqueness. This follows easily from the above estimates since the principal part of the system (3.2) is given by three decoupled vector fields. One can therefore integrate along characteristics. Equations (3.2)(a),(c) and the fact that X_{ϕ_2} is outgoing imply $\sigma_2^{n+1}=0$. $\square$

Next we show convergence of the iterates $\mathcal{V}^{0,n}$ to a short time solution of the nonlinear profile equations (3.1).

Proposition 3.6. *Consider the profile equations* (3.1)*, where* $G\in b\Gamma_T^m$*,* $m>\frac{d+2}{2}+1$*, and vanishes in* $t\leq 0$*. For some* $0<T_0\leq T$ *the system has a unique solution* $\mathcal{V}^0\in\Gamma_{T_0}^m$ *with* $\sigma_2=0$*.*

Proof. **1.** The iteration scheme (3.2) defines a sequence $(\mathcal{V}^{0,n})$ in Γ_T^m. Fixing $K>0$ we claim that for $T^*>0$ small enough,

$$|\mathcal{V}^{0,n}|_{m,T^*}+\langle\mathcal{V}^{0,n}\rangle_{m,T^*}<K \text{ for all } n. \tag{3.12}$$

Indeed, first observe that

$$|u|_{m,\gamma,T}\leq C_1\,|u|_{m,T}\leq C_2\,\mathrm{e}^{\gamma T}\,|u|_{m,\gamma,T},$$

and fix $\gamma>\gamma_0(K)$ such that $\sqrt{\gamma}\geq 2\,C(K)\,C_1$ for $\gamma_0(K)$ and $C(K)$ as in Proposition 3.5. Assuming (3.12) holds for $n\leq n_0$, we find that it holds for n_0+1 after shrinking T^* if necessary, using the estimate (3.8) and the fact that G vanishes in $t\leq 0$. This new choice of T^* works for all n.

2. Convergence of the iterates in $\Gamma_{T_0}^0$ to some $\mathcal{V}^0$ for a possibly smaller $T_0>0$ now follows from (3.12) by applying (3.8) when $m=0$ to the problem satisfied by $(\mathcal{V}^{0,n+1}-\mathcal{V}^{0,n})$. In view of (3.12) and a classical argument involving weak convergence and interpolation, we thereby obtain a solution $\mathcal{V}^0\in\Gamma_{T_0}^m$ with, in fact, a trace that lies in $b\Gamma_{T_0}^m$. This argument shows that the iterates $\mathcal{V}^{0,n}$ converge to $\mathcal{V}^0$ in $\Gamma_{T_0}^{m-1}$. $\square$

4 Error analysis

Next we carry out the error analysis sketched in section 1.4. In section 4.1 we define and derive estimates for moment-zero approximations. In section 4.2 we estimate interaction integrals involving both transversal and nontransversal interactions of pulses; these estimates are used later to estimate the first corrector $\mathcal{U}^1_{p,\varepsilon}$. Finally, in section 4.3 we complete the proof of Theorem 1.14 by proving the stronger result, Theorem 4.16.

4.1 Moment-zero approximations to $\mathcal{U}^0$

When constructing a corrector to the leading term in the approximate solution we must take primitives in θ of functions $\sigma(x,\theta)$ that decay to zero as $|\theta|\to\infty$. A difficulty is that such primitives do not necessarily decay to zero as $|\theta|\to\infty$, and this prevents us from using those primitives directly in the error analysis. The failure of the primitive to decay manifests itself on the Fourier transform side as a small divisor problem. To get around this difficulty we work with the primitive of a *moment-zero approximation* to σ, because such a primitive does have the desired decay.

We will use the following spaces:

Definition 4.1. *1.) For $s \geq 0$, we recall the notation* (1.6), *that is $E^s_T := \{U \in C(x_d, H^s_T(x', \theta_0)) \cap L^2(x_d, H^{s+1}_T(x', \theta_0))\}$. This space is equipped with the norm*

$$|U(x, \theta_0)|_{E^s_T} := |U|_{\infty,s,T} + |U|_{0,s+1,T}.$$

2.) Let $\mathcal{E}^s_T := \{\mathcal{U}(x, \theta_0, \xi_d) : |\mathcal{U}|_{\mathcal{E}^s_T} := \sup_{\xi_d \geq 0} |\mathcal{U}(\cdot, \cdot, \xi_d)|_{E^s_T} < \infty\}$.

Proposition 4.2. *For $s > (d+1)/2$ the spaces E^s_T and $\mathcal{E}^s_T$ are Banach algebras.*

Proof. This is a consequence of the Sobolev embedding Theorem and the fact that $L^\infty(b\Omega_T) \cap H^s(b\Omega_T)$ is a Banach algebra for $s \geq 0$. ☐

The proofs of the following two propositions follow directly from the definitions.

Proposition 4.3. *(a) For $s \geq 0$, let $\sigma(x, \theta) \in E^s_T$ and set $\tilde{\sigma}(x, \theta_0, \xi_d) := \sigma(x, \theta_0 + \omega\, \xi_d)$, $\omega \in \mathbb{R}$. Then $\tilde{\sigma} \in \mathcal{E}^s_T$ and*

$$|\tilde{\sigma}|_{\mathcal{E}^s_T} \leq C|\sigma|_{H^{s+1}_T}.$$

(b) For $\tilde{\sigma} \in \mathcal{E}^s_T$, set $\tilde{\sigma}_\varepsilon(x, \theta_0) := \tilde{\sigma}(x, \theta_0, \frac{x_d}{\varepsilon})$. Then

$$|\tilde{\sigma}_\varepsilon|_{E^s_T} \leq |\tilde{\sigma}|_{\mathcal{E}^s_T}.$$

Definition 4.4 (Moment-zero approximations)**.** *Let $0 < p < 1$, and let $\phi \in C^\infty(\mathbb{R})$ have* supp $\phi \subset$ *$\{m : |m| \leq 2\}$ with $\phi = 1$ on $\{|m| \leq 1\}$. Set $\phi_p(m) := \phi(\frac{m}{p})$ and $\chi_p := 1 - \phi_p$. For $\sigma(x, \theta) \in L^2(\Omega_T)$, define the* moment zero approximation *to σ, $\sigma_p(x, \theta)$ by*

$$\hat{\sigma}_p(x, m) := \chi_p(m)\, \hat{\sigma}(x, m), \tag{4.1}$$

where the hat denotes the Fourier transform in θ.

Proposition 4.5. *For $s \geq 1$ suppose $\sigma(x, \theta) \in \Gamma^{s+2}_T$, and define $\tilde{\sigma}(x, \theta_0, \xi_d) := \sigma(x, \theta_0 + \omega \xi_d)$. Then*

$$a)\ |\tilde{\sigma} - \tilde{\sigma}_p|_{\mathcal{E}^s_T} \leq C\, |\sigma|_{\Gamma^{s+2}_T} \sqrt{p},$$
$$b)\ |\partial_{x_d}\tilde{\sigma} - \partial_{x_d}\tilde{\sigma}_p|_{\mathcal{E}^{s-1}_T} \leq C\, |\sigma|_{\Gamma^{s+2}_T} \sqrt{p}.$$

Proof. **1.** Recall that $\sigma \in \Gamma^s_T \Leftrightarrow \theta^{\beta_1}\, \partial_x^{\beta_2}\, \partial_\theta^{\beta_3} \sigma(x, \theta) \in L^2(x, \theta)$ for $|\beta| \leq s$, which is also equivalent to $\partial_m^{\beta_1}\, \partial_x^{\beta_2}\, m^{\beta_3}\, \hat{\sigma}(x, m) \in L^2(x, m)$ for $|\beta| \leq s$. It follows that

$$\sigma \in \Gamma^{s+2}_T \Rightarrow \hat{\sigma}(x, m) \in H^{s+2}_T(x, m) \subset H^1(m, H^{s+1}(x)) \subset L^\infty(m, H^{s+1}(x)). \tag{4.2}$$

2. We have

$$\begin{aligned}
|\sigma - \sigma_p|^2_{H^{s+1}_T} &\sim \sum_{|\alpha|+k \leq s+1} |\partial_x^\alpha\, m^k\, \hat{\sigma}(x, m)\, (1 - \chi_p(m))|^2_{L^2(x,m)} \\
&= \sum_{|\alpha|+k \leq s+1} \int_{|m| \leq 2p} \int |\partial_x^\alpha\, m^k\, \hat{\sigma}(x, m)\, \phi_p(m)|^2\, \mathrm{d}x\, \mathrm{d}m \\
&\leq C \int_{|m| \leq 2p} |\hat{\sigma}(x, m)|^2_{H^{s+1}(x)}\, \mathrm{d}m \leq C\, |\sigma|^2_{\Gamma^{s+2}_T}\, (2p),
\end{aligned}$$

where the last inequality uses (4.2). The conclusion now follows from Propostion 4.3.

3. The proof of inequality *b)* in Proposition 4.5 is essentially the same. ☐

Proposition 4.6. *Let $\sigma(x,\theta) \in H^s_T$, $s \geq 0$, and let σ_p be a moment-zero approximation to σ. We have*

$$(a)\ |\sigma_p|_{H^s_T} \leq C\,|\sigma|_{H^s_T}\,,$$
$$(b)\ \ \textit{If } \sigma \in \Gamma^s_T,\ \textit{then } |\sigma_p|_{\Gamma^s_T} \leq \frac{C}{p^s}\,|\sigma|_{\Gamma^s_T}.$$

Proof. Part b) follows from (4.1). Indeed, for $|\beta| \leq s$,

$$|\partial_m^{\beta_1}\,\partial_x^{\beta_2}\,m^{\beta_3}\,\hat{\sigma}_p(x,m)|_{L^2_T} \leq \frac{C}{p^{\beta_1}}\,|\sigma|_{\Gamma^s_T}\,,$$

since $|\partial_m^{\beta_1}\chi_p| \leq C/p^{\beta_1}$. Taking $\beta_1 = 0$ we similarly obtain part a). □

Next we consider primitives of moment-zero approximations.

Proposition 4.7. *Let $\sigma(x,\theta) \in \Gamma^s_T$, $s > \frac{d}{2}+3$. Let $\sigma^*_p(x,\theta)$ be the unique primitive of σ_p in θ that decays to zero as $|\theta| \to \infty$. Then $\sigma^*_p \in \Gamma^s_T$ with moment zero, and*

$$\begin{aligned}(a)\ |\sigma^*_p|_{H^s_T} &\leq C\,\frac{|\sigma_p|_{H^s_T}}{p}\,,\\ (b)\ |\sigma^*_p|_{\Gamma^s_T} &\leq C\,\frac{|\sigma_p|_{\Gamma^s_T}}{p^{s+1}}\,.\end{aligned} \tag{4.3}$$

Proof. **1.** Since $\sigma_p(x,\theta) \in \Gamma^s_T$, $s > \frac{d}{2}+3$, we have $|\sigma_p(x,\theta)| \leq C\,\langle\theta\rangle^{-2}$ for all (x,θ). The unique θ-primitive of σ_p decaying to zero as $|\theta| \to \infty$ is thus

$$\sigma^*_p(x,\theta) = -\int_\theta^\infty \sigma_p(x,s)\,\mathrm{d}s = \int_{-\infty}^\theta \sigma_p(x,s)\,\mathrm{d}s.$$

Moreover, we have

$$\partial_\theta\sigma^*_p = \sigma_p \Rightarrow im\,\widehat{\sigma^*_p} = \widehat{\sigma_p} = \chi_p\,\hat{\sigma}, \qquad \text{so } \widehat{\sigma^*_p} = \frac{\chi_p}{im}\,\hat{\sigma}. \tag{4.4}$$

Since $|m| \geq p$ on the support of χ_p, this gives

$$|\widehat{\sigma^*_p}(x,m)| \leq C\,\frac{|\hat{\sigma}(x,m)|}{p} \tag{4.5}$$

and (4.3)(a) follows directly from this. From (4.4) we also obtain $\widehat{\sigma^*_p}(x,0) = 0$.

2. The proof of (4.3) (b) is almost the same, except that now one uses

$$\left|\partial_m^s\left(\frac{\chi_p}{m}\right)\right| \leq \frac{C}{p^{s+1}}.$$

□

Proposition 4.8. *Let $\sigma(x,\theta)$ and $\tau(x,\theta)$ belong to H^s_T, $s > \frac{d+2}{2}$. Then*

$$|\sigma\,\tau - (\sigma\,\tau)_p|_{H^s_T} \leq C\,|\sigma|_{H^s_T}\,|\tau|_{H^s_T}\,\sqrt{p}. \tag{4.6}$$

Proof. With $*$ denoting convolution in m we have

$$\begin{aligned}|\sigma\,\tau-(\sigma\,\tau)_p|^2_{H^s_T} &\sim \sum_{|\alpha|+k\le s+1}|\partial^\alpha_x\,m^k\,(\hat\sigma*\hat\tau)(x,m)\,(1-\chi_p(m))|^2_{L^2(x,m)}\\ &\le C\int_{|m|\le 2p}|(\hat\sigma*\hat\tau)(x,m)|^2_{H^s(x)}\,\mathrm{d}m\\ &\le C\int_{|m|\le 2p}\Big(\int|\hat\sigma(x,m-m_1)|_{H^s(x)}\,|\hat\tau(x,m_1)|_{H^s(x)}\,\mathrm{d}m_1\Big)^2\,\mathrm{d}m\\ &\le C\,p\,|\hat\sigma(x,m)|^2_{L^2(m,H^s(x))}\,|\hat\tau(x,m)|^2_{L^2(m,H^s(x))}\le C\,p\,|\sigma|^2_{H^s_T}\,|\tau|^2_{H^s_T}.\end{aligned}$$

□

Proposition 4.9. *Let $\sigma(x,\theta)$ and $\tau(x,\theta)$ belong to Γ^s_T, $s>\frac{d}{2}+3$ and let $(\sigma\tau)^*_p$ denote the unique primitive of $(\sigma\tau)_p$ that decays to zero as $|\theta|\to\infty$. Then*

$$|(\sigma\,\tau)^*_p|_{H^s_T}\le C\,\frac{|\sigma|_{H^s_T}\,|\tau|_{H^s_T}}{p}.$$

Proof. Since Γ^s_T is a Banach algebra, Proposition 4.7 implies $(\sigma\tau)^*_p\in\Gamma^s_T$ with moment zero and

$$|(\sigma\,\tau)^*_p|_{H^s_T}\le C\,\frac{|(\sigma\,\tau)_p|_{H^s_T}}{p}.$$

Since H^s_T is a Banach algebra, the result now follows from Proposition 4.6(a). □

4.2 Estimates of interaction integrals

Pulses do not interact to produce resonances that affect the leading order profiles as in the wavetrain case. However, interaction integrals must be estimated carefully in order to do the error analysis.

The following propositions will be used in the error analysis for estimating terms related to $\mathcal{U}^1$ as in (1.30), where the $\mathcal{F}_i$ appearing there are given by (1.20); in particular, we must estimate primitives of products of pulses. In some of the estimates below we must introduce moment-zero approximations to avoid errors that are too large to be useful in the error analysis. We begin with an estimate of "transversal interactions".

Proposition 4.10. *Let t be the smallest integer greater than $\frac{d}{2}+3$ and let $s\ge 0$. Let $\sigma_1(x,\theta)$, $\sigma_2(x,\theta)$ belong to $\Gamma^t_T\cap H^{s+1}_T$ and define*

$$u(x,\theta_0,\xi_d):=\int_\infty^{\xi_d}\sigma_1(x,\theta_0+\omega\,\xi_d+\alpha\,s)\,\sigma_2(x,\theta_0+\omega\,\xi_d+s)\,\mathrm{d}s, \tag{4.7}$$

where ω, α are real and $\alpha\notin\{0,1\}$. With $u_\varepsilon(x,\theta_0):=u(x,\theta_0,\frac{x_d}{\varepsilon})$ we have

$$|u_\varepsilon|_{E^s_T}\le C\,(|\sigma_1|_{H^{s+1}_T}\,|\sigma_2|_{\Gamma^t_T}+|\sigma_2|_{H^{s+1}_T}\,|\sigma_1|_{\Gamma^t_T}).$$

uniformly for $\varepsilon\in(0,1]$.

Proof. **1.** For fixed x_d and ε we first estimate

(4.8)
$$\left|\int_\infty^{x_d/\varepsilon} \sigma_1\,\sigma_2\,\mathrm{d}s\right|_{H^s_T(x',\theta_0)} \sim \sum_{|\alpha|\le s}\left|\int_\infty^{x_d/\varepsilon} \partial^\alpha_{x'}\,(\sigma_1\,\sigma_2)\,\mathrm{d}s\right|_{L^2(x',\theta_0)} + \sum_{k\le s}\left|\int_\infty^{x_d/\varepsilon} \partial^k_{\theta_0}\,(\sigma_1\,\sigma_2)\,\mathrm{d}s\right|_{L^2(x',\theta_0)}$$
$$:= A_s(x_d) + B_s(x_d).$$

Here and below σ_1, σ_2 and their derivatives are evaluated at the points indicated in (4.7) with $\xi_d = \frac{x_d}{\varepsilon}$, unless explicitly stated otherwise.

2. To estimate $A_s(x_d)$ we consider for $|\alpha_1| + |\alpha_2| = |\alpha|$:

$$\left|\int_\infty^{x_d/\varepsilon} \partial^{\alpha_1}_{x'}\sigma_1\,\partial^{\alpha_2}_{x'}\sigma_2\,\mathrm{d}s\right|_{L^2(x',\theta_0)} \le \left|\int_\infty^{x_d/\varepsilon} \left|\partial^{\alpha_1}_{x'}\sigma_1\,\partial^{\alpha_2}_{x'}\sigma_2\right|_{L^2(x')}\,\mathrm{d}s\right|_{L^2(\theta_0)}$$
$$\le \left|\int_{-\infty}^{\infty} (|\sigma_1|_{L^\infty(x')}\,|\sigma_2|_{H^s(x')} + |\sigma_1|_{H^s(x')}\,|\sigma_2|_{L^\infty(x')})\,\mathrm{d}s\right|_{L^2(\theta_0)} \le A_{1,s}(x_d) + A_{2,s}(x_d),$$

where we have used a Moser estimate in the x' variable. Setting $z = \theta_0 + \omega\,\frac{x_d}{\varepsilon} + \alpha\,s$, we obtain

$$A_{1,s}(x_d) = C\left|\int_{-\infty}^{\infty} |\sigma_1(x,z)|_{L^\infty(x')}\left|\sigma_2\left(x, (\theta_0 + \omega\frac{x_d}{\varepsilon})(1-\frac{1}{\alpha}) + \frac{z}{\alpha}\right)\right|_{H^s(x')}\mathrm{d}z\right|_{L^2(\theta_0)}$$
$$\le C\int_{-\infty}^{\infty} |\sigma_1(x,z)|_{L^\infty(x')}\,|\sigma_2|_{L^2(\theta,H^s(x'))}\,\mathrm{d}z$$
$$= C\,|\sigma_2|_{L^2(\theta,H^s(x'))}\int_{-\infty}^{\infty} |\sigma_1(x,z)|_{L^\infty(x')}\,\langle z\rangle^2\,\frac{\mathrm{d}z}{\langle z\rangle^{-2}}$$
$$\le C\,|\sigma_2|_{L^2(\theta,H^s(x'))}\,|\sigma_1(x,z)\langle z\rangle^2|_{L^\infty(x,z)} \le C\,|\sigma_2|_{H^s_T(x',\theta)}\,|\sigma_1|_{\Gamma^t_T}, \tag{4.9}$$

where the last inequality uses Remark 3.4. The estimate of $A_{2,s}(x_d)$ is similar.

3. Recalling the definition of the E^s_T norm and using

$$|\sigma_2|_{C(x_d,H^s_T(x',\theta))} \le C\,|\sigma_2|_{H^1(x_d,H^s_T(x',\theta))} \le C\,|\sigma_2|_{H^{s+1}_T(x,\theta)}\,,$$

we obtain from (4.9):

(4.10)
$$|A_{1,s}(x_d)\,A_{2,s}(x_d)|_{C(x_d)} + |A_{1,s+1}(x_d)\,A_{2,s+1}(x_d)|_{L^2(x_d)} \le C\,(|\sigma_1|_{H^{s+1}_T}\,|\sigma_2|_{\Gamma^t_T} + |\sigma_2|_{H^{s+1}_T}\,|\sigma_1|_{\Gamma^t_T}).$$

4. To estimate $B_s(x_d)$ in (4.8) we consider for $k_1 + k_2 = k$:

$$\int_\infty^{x_d/\varepsilon} \partial^{k_1}_{\theta_0}\sigma_1\,\partial^{k_2}_{\theta_0}\sigma_2\,\mathrm{d}s = \pm\int_\infty^{x_d/\varepsilon} (\partial^k_{\theta_0}\sigma_1)\,\sigma_2 ds + (\text{boundary terms}).$$

Each boundary term has the form $\partial^{m_1}_{\theta_0}\sigma_1\,\partial^{m_2}_{\theta_0}\sigma_2$, $m_1 + m_2 < k$, where s is evaluated at x_d/ε. We estimate such terms using Moser estimates as follows:

$$|\partial^{m_1}_{\theta_0}\sigma_1\,\partial^{m_2}_{\theta_0}\sigma_2|_{L^2(x',\theta_0)} \le C\,(|\sigma_1|_{L^\infty(x,\theta)}\,|\sigma_2|_{H^{s-1}(x',\theta_0)} + |\sigma_1|_{H^{s-1}(x',\theta_0)}\,|\sigma_2|_{L^\infty(x,\theta_0)}). \tag{4.11}$$

For the integral term setting $z = \theta_0 + \omega \frac{x_d}{\varepsilon} + s$, we have

(4.12)
$$\left| \int_{\infty}^{x_d/\varepsilon} (\partial_{\theta_0}^k \sigma_1)\, \sigma_2 \, \mathrm{d}s \right|_{L^2(x',\theta_0)} \leq C \left| \int_{-\infty}^{\infty} |\partial_{\theta_0}^k \sigma_1 \Big(x, (\theta_0 + \omega \frac{x_d}{\varepsilon})(1-\alpha) + \alpha z \Big)\, \sigma_2(x,z)|_{L^2(x')} \, \mathrm{d}z \right|_{L^2(\theta_0)}$$
$$\leq C \left| \int_{-\infty}^{\infty} |\partial_{\theta_0}^k \sigma_1 \Big(x, (\theta_0 + \omega \frac{x_d}{\varepsilon})(1-\alpha) + \alpha z \Big)\, |_{L^2(x')}\, |\sigma_2(x,z)|_{L^\infty(x)} \, \mathrm{d}z \right|_{L^2(\theta_0)}$$
$$\leq C\, |\sigma_1|_{L^2(x', H^s(\theta))}\, |\sigma_2(x,z)\langle z\rangle^2|_{L^\infty(x,z)} \leq C\, |\sigma_1|_{H^s_T(x',\theta)}\, |\sigma_2|_{\Gamma^t_T}.$$

From (4.11) and (4.12), we obtain parallel to (4.10):

$$|B_s(x_d)|_{C(x_d)} + |B_{s+1}(x_d)|_{L^2(x_d)} \leq C\, (|\sigma_1|_{H^{s+1}_T}\, |\sigma_2|_{\Gamma^t_T} + |\sigma_2|_{H^{s+1}_T}\, |\sigma_1|_{\Gamma^t_T}),$$

completing the proof. □

The previous estimate of transversal interactions did not require the use of moment-zero approximations. However, nontransversal interactions of pulses can produce errors that are too big to be helpful in the error analysis. Thus, we are forced to use a moment-zero approximation in the next proposition.

Proposition 4.11. *Let $\sigma(x,\theta)$ and $\tau(x,\theta)$ belong to Γ^s_T, $s > \frac{d}{2} + 3$. For $\alpha, \omega \in \mathbb{R}$, $\alpha \neq 0$ set*

$$f(x,\theta_0,\xi_d) := \int_{\infty}^{\xi_d} (\sigma\,\tau)_p(x, \theta_0 + \omega\,\xi_d + \alpha\, s)\, \mathrm{d}s.$$

Then

$$\left| f(x,\theta_0,\frac{x_d}{\varepsilon}) \right|_{E^{s-1}_T} \leq C\, \frac{|\sigma|_{H^s_T}\, |\tau|_{H^s_T}}{p}\,.$$

Proof. The integral equals $\alpha^{-1}\, (\sigma\,\tau)^*_p(x, \theta_0 + \xi_d(\omega + \alpha))$ so the estimate follows first by applying Proposition 4.9 and then by applying Proposition 4.3. □

Corollary 4.12. *Let $\sigma(x,\theta)$, $\tau(x,\theta)$, and ω, α be as in Proposition 4.11 and set*

$$g(x,\theta_0,\xi_d) := \int_{\infty}^{\xi_d} (\sigma_p\, \tau_p)_p(x, \theta_0 + \omega\,\xi_d + \alpha\, s)\, \mathrm{d}s.$$

Then

$$\left| g(x,\theta_0,\frac{x_d}{\varepsilon}) \right|_{E^{s-1}_T} \leq C\, \frac{|\sigma|_{H^s_T}\, |\tau|_{H^s_T}}{p}\,.$$

Proof. First apply Proposition 4.11 and then Proposition 4.6(a). □

Proposition 4.13. *For $s > \frac{d}{2} + 3$ let $\sigma(x,\theta) \in H^s_T$, $\tau(x,\theta) \in \Gamma^{s+1}_T$. With $\omega, \alpha \in \mathbb{R}$, $\alpha \neq 0$ set*

$$h(x,\theta_0,\xi_d) := \sigma(x, \theta_0 + \omega\,\xi_d) \int_{\infty}^{\xi_d} \partial_{\theta_0}\tau(x, \theta_0 + \omega\,\xi_d + \alpha\, s)\, \mathrm{d}s.$$

Then

$$\left| h(x,\theta_0,\frac{x_d}{\varepsilon}) \right|_{E^{s-1}_T} \leq C\, |\sigma|_{H^s_T}\, |\tau|_{H^s_T}\,.$$

Proof. The integral is equal to $\alpha^{-1}\,\tau(x,\theta_0+\xi_d(\omega+\alpha))$ so the estimate follows from the fact that E^{s-1}_T is a Banach algebra together with Proposition 4.3. $\square$

In the next Proposition we must use a moment-zero approximation since $\tau(x,\theta)$ may not have moment zero.

Proposition 4.14. *For $s>\frac{d}{2}+3$ let $\sigma(x,\theta)\in H^s_T$, $\tau(x,\theta)\in\Gamma^s_T$. With $\omega,\alpha\in\mathbb{R}$, $\alpha\neq 0$ set*

$$j(x,\theta_0,\xi_d):=\partial_{\theta_0}\sigma(x,\theta_0+\omega\,\xi_d)\int_\infty^{\xi_d}\tau_p(x,\theta_0+\omega\,\xi_d+\alpha\,s)\,\mathrm{d}s.$$

Then

$$\left|j(x,\theta_0,\frac{x_d}{\varepsilon})\right|_{E^{s-2}_T}\leq C\,\frac{|\sigma|_{H^s_T}\,|\tau|_{H^{s-1}_T}}{p}\,.$$

Proof. The integral is equal to $\alpha^{-1}\,\tau^*_p(x,\theta_0+\xi_d(\omega+\alpha))$. The estimate follows by the argument of Proposition 4.13, except that now we also need Proposition 4.7(a) and Proposition 4.6(a). $\square$

The proof of the next Proposition is evident from the proof of Proposition 4.14.

Proposition 4.15. *For $s>\frac{d}{2}+3$ and $\omega,\alpha\in\mathbb{R}$, $\alpha\neq 0$, let $\sigma\in\Gamma^s_T$ and set*

$$k(x,\theta_0,\xi_d)=\int_\infty^{\xi_d}\sigma_p(x,\theta_0+\omega\xi_d+\alpha s)ds.$$

Then

$$\left|k(x,\theta_0,\frac{x_d}{\varepsilon})\right|_{E^{s-1}_T}\leq C\,\frac{|\sigma|_{H^s_T}}{p}\,.$$

4.3 Proof of Theorem 1.14

Now we are ready to prove Theorem 1.14, which shows that the approximate solution $u^a_\varepsilon(x)$ converges in L^∞ to the exact solution u_ε of Theorem 1.12 as $\varepsilon\to 0$. In this section we prove the following more precise Theorem, which implies Theorem 1.14 as an immediate corollary. As before we focus on the 3×3 strictly hyperbolic case to ease the exposition. The mostly minor changes needed to treat $N\times N$ systems satisfying Assumptions 1.1, 1.2, and 1.6 are described in section 5.

Theorem 4.16. *For $M_0=3d+5$ and $s\geq 1+[M_0+\frac{d+1}{2}]$, let $G(x',\theta_0)\in b\Gamma^{s+1}_T$ and suppose $G=0$ in $t\leq 0$. Let $U_\varepsilon(x,\theta_0)\in E^s_{T_0}$ be the exact solution to the singular system* (1.5) *for $0<\varepsilon\leq\varepsilon_0$ given by Theorem 1.12, let $\mathcal{V}^0=(\sigma_1,\sigma_2,\sigma_3)\in\Gamma^{s+1}_{T_0}$ be the profile given by Proposition 3.6, and let $\mathcal{U}^0\in\mathcal{E}^s_{T_0}$ be defined by*

$$\mathcal{U}^0(x,\theta_0,\xi_d):=\sum_{j=1}^3\sigma_j(x,\theta_0+\omega_j\,\xi_d)\,r_j.$$

Here $0<T_0\leq T$ is the minimum of the existence times for the quasilinear problems (1.5) *and* (1.27)*. Define*

$$\mathcal{U}^0_\varepsilon(x,\theta_0):=\mathcal{U}^0(x,\theta_0,\frac{x_d}{\varepsilon}).$$

The family $\mathcal{U}^0_\varepsilon$ *is uniformly bounded in* $E^s_{T_0}$ *for* $0<\varepsilon\leq\varepsilon_0$*; moreover, there exists* $0<T_1\leq T_0$ *and* $C>0$ *such that*

$$|U_\varepsilon-\mathcal{U}^0_\varepsilon|_{E^{s-3}_{T_1}}\leq C\varepsilon^{\frac{1}{2M_1+5}},\tag{4.13}$$

where M_1 *is the smallest integer* $>\frac{d}{2}+3$.

The proof of Theorem 4.16 will use the strategy of *simultaneous Picard iteration* first used by Joly, Métivier, and Rauch in [11] to justify leading term expansions for initial value problems on domains without boundary. Consider the iteration schemes for the quasilinear problems (1.5) and (1.27):

$$\begin{aligned}&a)\ \partial_{x_d}U^{n+1}_\varepsilon+\sum_{j=0}^{d-1}\tilde{A}_j(\varepsilon U^n_\varepsilon)\left(\partial_{x_j}+\frac{\beta_j\partial_{\theta_0}}{\varepsilon}\right)U^{n+1}_\varepsilon=F(\varepsilon U^n_\varepsilon)U^n_\varepsilon,\\&b)\ B(\varepsilon U^n_\varepsilon)\,U^{n+1}_\varepsilon|_{x_d=0}=G(x',\theta_0),\\&c)\ U^{n+1}_\varepsilon=0\text{ in }t<0,\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}&a)\ \mathbf{E}\mathcal{U}^{0,n+1}=\mathcal{U}^{0,n+1}\\&b)\ \mathbf{E}\left(\tilde{L}(\partial)\mathcal{U}^{0,n+1}+M(\mathcal{U}^{0,n},\partial_{\theta_0}\mathcal{U}^{0,n+1})\right)=\mathbf{E}\left(F(0)\mathcal{U}^{0,n}\right)\\&c)\ B(0)\,\mathcal{U}^{0,n+1}|_{x_d=0,\xi_d=0}=G(x',\theta_0)\\&d)\ \mathcal{U}^{0,n+1}=0\text{ in }t<0,\end{aligned}\tag{4.15}$$

where $\mathcal{U}^{0,n}(x,\theta_0,\xi_d):=\sum_{j=1}^3\sigma^n_j(x,\theta_0+\omega_j\,\xi_d)\,r_j$ for σ^n_j as constructed in Proposition 3.5. Setting

$$\mathcal{U}^{0,n}_\varepsilon(x,\theta_0):=\mathcal{U}^{0,n}(x,\theta_0,\frac{x_d}{\varepsilon}),$$

we observe that to prove the theorem it suffices to prove boundedness of the family $\mathcal{U}^0_\varepsilon$ in $E^s_{T_0}$ along with the following three statements:

(4.16)

(a) $\lim_{n\to\infty}U^n_\varepsilon=U_\varepsilon$ in $E^{s-1}_{T_0}$ uniformly with respect to $\varepsilon\in(0,\varepsilon_0]$

(b) $\lim_{n\to\infty}\mathcal{U}^{0,n}_\varepsilon=\mathcal{U}^0_\varepsilon$ in $E^{s-1}_{T_0}$ uniformly with respect to $\varepsilon\in(0,\varepsilon_0]$

(c) There exist positive constants $T_1\leq T_0$ and C_1, independent of n, such that for every n

$$|U^n_\varepsilon-\mathcal{U}^{0,n}_\varepsilon|_{E^{s-3}_{T_1}}\leq C_1\,\varepsilon^{\frac{1}{2M_1+5}}.$$

The first statement, together with uniform boundedness of the families U^n_ε, U_ε in $E^s_{T_0}$, is proved in Theorem 2.6 by showing convergence of the scheme (4.14) using the following linear estimate (which is a consequence of Proposition 2.4).

Proposition 4.17. *Let* $s\geq[M_0+\frac{d+1}{2}]$ *and consider the problem* (4.14)*, where* $G\in H^{s+1}$ *vanishes in* $t\leq0$*, and where the right side of* (4.14)*(a) is replaced by* $\mathcal{F}\in E^s_T$*. Suppose* $U^n_\varepsilon\in E^s_T$ *and that for some* $K>0$, $\varepsilon_1>0$*, we have*

$$|U^n_\varepsilon|_{E^s_T}+|\varepsilon\,\partial_{x_d}U^n_\varepsilon|_{L^\infty}\leq K\ \textit{for}\ \varepsilon\in(0,\varepsilon_1].$$

Then there exist constants $T_0(K)$ *and* $\varepsilon_0(K)\leq\varepsilon_1$ *such that for* $0<\varepsilon\leq\varepsilon_0$ *and* $T\leq T_0$ *we have*

$$|U^{n+1}_\varepsilon|_{E^s_T}+\sqrt{T}\,\langle U^{n+1}_\varepsilon|_{x_d=0}\rangle_{s+1,T}\leq C(K)\,\left(T\,|\mathcal{F}|_{E^s_T}+\sqrt{T}\,\langle G\rangle_{s+1,T}\right).$$

Proof of Theorem 4.16. **1.** The boundedness of $\mathcal{U}^0_\varepsilon$ in $E^s_{T_0}$ and (4.16)(b) follow directly from Proposition 4.3, together with the fact that $\mathcal{V}^0 \in H^{s+1}_{T_0}$ and $\mathcal{V}^{0,n} \to \mathcal{V}^0$ in $H^s_{T_0}$. (In fact, the proof of Proposition 3.6 shows that the $\mathcal{V}^{0,n}$ are bounded in $\Gamma^{s+1}_{T_0}$ and $\mathcal{V}^{0,n} \to \mathcal{V}^0$ in $\Gamma^s_{T_0}$.)

2. The approximate solution $\mathcal{U}^{0,n}_\varepsilon$ is by itself too crude; in order to prove (4.13) using Proposition 4.17 we must construct a corrector $\varepsilon\,\mathcal{U}^1_{p,\varepsilon}$ that lies in some $E^r_{T_0}$ space. To achieve this we first approximate $\mathcal{U}^{0,n}$ and $\mathcal{U}^{0,n+1}$ by moment-zero approximations $\mathcal{U}^{0,n}_p$ and $\mathcal{U}^{0,n+1}_p$. For now we fix $0 < p < 1$ and define for each n

$$\mathcal{U}^{0,n}_p(x,\theta_0,\xi_d) = \sum_{j=1}^{3} \sigma^n_{j,p}(x,\theta_0 + \omega_j\xi_d)\, r_j,$$

where $\sigma^n_{j,p}$ is the moment-zero approximation to σ^n_j defined by (4.1). Thus we have $\mathcal{U}^{0,n}_p(x,\theta_0,\xi_d) = \mathbf{E}\,\mathcal{U}^{0,n}_p(x,\theta_0,\xi_d)$ and by Proposition 4.5

$$|\mathcal{U}^{0,n} - \mathcal{U}^{0,n}_p|_{\mathcal{E}^{s-1}_{T_0}} \le C\,\sqrt{p}\,, \qquad \text{and} \qquad |\partial_{x_d}\mathcal{U}^{0,n+1} - \partial_{x_d}\mathcal{U}^{0,n+1}_p|_{\mathcal{E}^{s-2}_{T_0}} \le C\,\sqrt{p}\,, \tag{4.17}$$

for C independent of n[15].

3. For now we express the induction assumption as: there exists $0 < a < 1$ (to be determined) and positive constants C_1, $T_1 \le T_0$ such that

$$|U^n_\varepsilon - \mathcal{U}^{0,n}_\varepsilon|_{E^{s-3}_{T_1}} \le C_1\,\varepsilon^a. \tag{4.18}$$

The boundedness of the family U^n_ε in $E^s_{T_0}$ together with (4.18) imply

$$|F(\varepsilon U^n_\varepsilon)\,U^n_\varepsilon - F(0)\,\mathcal{U}^{0,n}_\varepsilon|_{E^{s-3}_{T_0}} \le C\,\varepsilon^a.$$

In view of (4.17) and Proposition 4.3 this implies

$$|F(\varepsilon U^n_\varepsilon)\,U^n_\varepsilon - F(0)\,\mathcal{U}^{0,n}_{p,\varepsilon}|_{E^{s-3}_{T_0}} \le C\,(\sqrt{p} + \varepsilon^a). \tag{4.19}$$

4. Define

$$\mathcal{G}_p := \tilde{L}(\partial_x)\mathcal{U}^{0,n+1}_p + M(\mathcal{U}^{0,n}_p, \partial_{\theta_0}\mathcal{U}^{0,n+1}_p).$$

We claim that

$$|\mathbf{E}\mathcal{G}_p - \mathbf{E}(F(0)\,\mathcal{U}^{0,n}_p)|_{\mathcal{E}^{s-2}_{T_0}} \le C\sqrt{p}. \tag{4.20}$$

Indeed, from (4.17) and the explicit formula (1.25) for the action of $\mathbf{E}$ on functions of type $\mathcal{F}$, we have

$$|\mathbf{E}\left(F(0)\,\mathcal{U}^{0,n} - F(0)\,\mathcal{U}^{0,n}_p\right)|_{\mathcal{E}^{s-1}_{T_0}} \le C\sqrt{p}.$$

But $\mathbf{E}(F(0)\mathcal{U}^{0,n})$ is given by the left side of (4.15)(b), so (4.20) follows by observing that (4.17) and Proposition 4.2 imply

$$\begin{aligned}
&\left|\mathbf{E}\left(\tilde{L}(\partial_x)\left(\mathcal{U}^{0,n+1} - \mathcal{U}^{0,n+1}_p\right)\right)\right|_{\mathcal{E}^{s-2}_{T_0}} \le C\sqrt{p}\\
&|\mathbf{E}\left(M(\mathcal{U}^{0,n}, \partial_{\theta_0}\mathcal{U}^{0,n+1}) - M(\mathcal{U}^{0,n}_p, \partial_{\theta_0}\mathcal{U}^{0,n+1}_p)\right)|_{\mathcal{E}^{s-2}_{T_0}} \le C\sqrt{p}.
\end{aligned} \tag{4.21}$$

[15] Constants $C, C_1 ...$ appearing in this proof are all independent of n and ε.

Here we have used the fact that the arguments of $\mathbf{E}$ in (4.21) are functions of type $\mathcal{F}$, so the formula (1.25) can be applied.

5. Next define the operator

$$\mathbb{L}_0 := \tilde{L}(\partial_x) + \frac{1}{\varepsilon}\tilde{L}(\mathrm{d}\phi_0)\partial_{\theta_0} + M(\mathcal{U}^{0,n}_{p,\varepsilon}, \partial_{\theta_0}),$$

which is an approximation to the operator appearing on the left side of (4.14)(a) that will allow us to use Proposition 1.21 to construct a useful corrector $\mathcal{U}^1_p$. Indeed, we claim

$$|\mathbb{L}_0 U^{n+1}_\varepsilon - F(\varepsilon U^n_\varepsilon)U^n_\varepsilon|_{E^{s-3}_{T_0}} \leq C(\sqrt{p} + \varepsilon^a). \tag{4.22}$$

This follows from (4.14)(a) and the estimates

$$\begin{aligned}
&|\tilde{A}_j(\varepsilon U^n_\varepsilon)\partial_{x_j} U^{n+1}_\varepsilon - \tilde{A}_j(0)\partial_{x_j} U^{n+1}_\varepsilon|_{E^{s-1}_{T_0}} \leq C\varepsilon\,,\\
&\left|\frac{1}{\varepsilon}\,\tilde{A}_j(\varepsilon U^n_\varepsilon)\,\beta_j\,\partial_{\theta_0} U^{n+1}_\varepsilon - \left(\frac{1}{\varepsilon}\,\tilde{A}_j(0)\,\beta_j\,\partial_{\theta_0} U^{n+1}_\varepsilon + \mathrm{d}\tilde{A}_j(0)\cdot U^n_\varepsilon\,\beta_j\,\partial_{\theta_0} U^{n+1}_\varepsilon\right)\right|_{E^{s-1}_{T_0}} \leq C\,\varepsilon\,,\\
&\left|\mathrm{d}\tilde{A}_j(0)\cdot(U^n_\varepsilon - \mathcal{U}^{0,n}_{p,\varepsilon})\,\beta_j\,\partial_{\theta_0} U^{n+1}_\varepsilon\right|_{E^{s-3}_{T_0}} \leq C\,|U^n_\varepsilon - \mathcal{U}^{0,n}_{p,\varepsilon}|_{E^{s-3}_{T_0}} \leq C\,(\sqrt{p} + \varepsilon^a).
\end{aligned} \tag{4.23}$$

6. Construction of the corrector. First observe that since $\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})\mathcal{U}^{0,n+1}_p = 0$, we have

$$\mathbb{L}_0\,\mathcal{U}^{0,n+1}_{p,\varepsilon} = \mathcal{G}_{p,\varepsilon}\,,$$

and thus

$$\begin{aligned}
&\mathbb{L}_0\mathcal{U}^{0,n+1}_{p,\varepsilon} - F(0)\mathcal{U}^{0,n}_{p,\varepsilon} = \mathcal{G}_{p,\varepsilon} - F(0)\mathcal{U}^{0,n}_{p,\varepsilon} =\\
&\big(\mathbf{E}(\mathcal{G}_p - F(0)\mathcal{U}^{0,n}_p)\big)_\varepsilon + \big((I - \mathbf{E})(\mathcal{G}_p - F(0)\mathcal{U}^{0,n}_p)\big)_\varepsilon\,.
\end{aligned} \tag{4.24}$$

We have

$$|\,\big(\mathbf{E}(\mathcal{G}_p - F(0)\mathcal{U}^{0,n}_p)\big)_\varepsilon\,|_{E^{s-2}_{T_0}} \leq C\,\sqrt{p}\,, \tag{4.25}$$

by (4.20), the formula (1.25) for $\mathbf{E}$, and Proposition 4.3. The second term on the right in (4.24) is not small, so we construct $\mathcal{U}^1_p$ to solve (most of) it away. By Proposition 1.21 the function $\tilde{\mathcal{U}}^1_p := -\mathbf{R}_\infty\left((I - \mathbf{E})(\mathcal{G}_p - F(0)\mathcal{U}^{0,n}_p)\right)$ satisfies

$$\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})\tilde{\mathcal{U}}^1_p = -(I - \mathbf{E})(\mathcal{G}_p - F(0)\mathcal{U}^{0,n}_p). \tag{4.26}$$

However, this choice of $\tilde{\mathcal{U}}^1_p$ is too large to be useful in the error analysis.

7. To remedy this problem we replace $(I - \mathbf{E})\mathcal{G}_p$ by a modification $[(I - \mathbf{E})\mathcal{G}_p]_{mod}$ defined as follows. First, using (1.19) and Remark 1.19 we have

$$(I - \mathbf{E})\mathcal{G}_p = \sum_{i=1}^{3}\left(-\sum_{k\neq i} V^i_k \sigma^{n+1}_{k,p} + \sum_{k\neq i} c^i_k \sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p} + \sum_{l\neq m} d^i_{l,m}\sigma^n_{l,p}\partial_{\theta_0}\sigma^{n+1}_{m,p}\right) r_i, \tag{4.27}$$

where $\sigma^n_{q,p} = \sigma^n_{q,p}(x, \theta_0 + \omega_q\xi_d)$. The problem is caused by the nontransversal interaction terms given by the middle sum over $k \neq i$, so we define

$$[(I - \mathbf{E})\mathcal{G}_p]_{mod} = \sum_{i=1}^{3}\left(-\sum_{k\neq i} V^i_k \sigma^{n+1}_{k,p} + \sum_{k\neq i} c^i_k (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p})_p + \sum_{l\neq m} d^i_{l,m}\sigma^n_{l,p}\partial_{\theta_0}\sigma^{n+1}_{m,p}\right) r_i, \tag{4.28}$$

and we set

$$\mathcal{U}_p^1 := -\mathbf{R}_\infty \left([(I-\mathbf{E})\mathcal{G}_p]_{mod} - (I-\mathbf{E})F(0)\mathcal{U}_p^{0,n})\right). \tag{4.29}$$

Instead of (4.26) we have

$$\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})\mathcal{U}_p^1 = -[(I-\mathbf{E})\mathcal{G}_p]_{mod} + (I-\mathbf{E})F(0)\mathcal{U}_p^{0,n}. \tag{4.30}$$

For later use we set

$$D(x,\theta_0,\xi_d) := (I-\mathbf{E})\mathcal{G}_p - [(I-\mathbf{E})\mathcal{G}_p]_{mod}$$

and estimate

$$|D(x,\theta_0,\frac{x_d}{\varepsilon})|_{E_T^{s-3}} \leq C\sqrt{p}. \tag{4.31}$$

Indeed, using Propositions 4.8 and 4.6(a) we have

$$\begin{aligned}
&|\left(\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p} - (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p})_p\right)(x,\theta_0+\omega_k\frac{x_d}{\varepsilon})|_{E_T^{s-3}}\\
&\qquad\leq |\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p} - \sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p})_p|_{H_T^{s-2}} \leq |\sigma^n_k|_{H_T^{s-2}}|\sigma^{n+1}_k|_{H_T^{s-1}}\sqrt{p}.
\end{aligned}$$

8. Estimate of $|\mathcal{U}^1_{p,\varepsilon}|_{E_T^{s-2}}$. By (4.29),(4.28) and the formula (1.26) for $\mathbf{R}_\infty$, for $i=1,2,3$ we must estimate $|b_i(x,\theta_0,\frac{x_d}{\varepsilon})|_{E_T^{s-2}}$, where $b_i(x,\theta_0,\xi_d) =$

(4.32)

$$\begin{aligned}
&\sum_{k\neq i} c^i_k \int_\infty^{\xi_d} \left(\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p}\right)_p (x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))\,\mathrm{d}s+\\
&\sum_{m\neq i} d^i_{i,m}\int_\infty^{\xi_d} \sigma^n_{i,p}(x,\theta_0+\omega_i\xi_d)\partial_{\theta_0}\sigma^{n+1}_{m,p}(x,\theta_0+\omega_i\xi_d+s(\omega_m-\omega_i))\,\mathrm{d}s+\\
&\sum_{l\neq i} d^i_{l,i}\int_\infty^{\xi_d} \sigma^n_{l,p}(x,\theta_0+\omega_i\xi_d+s(\omega_l-\omega_i))\partial_{\theta_0}\sigma^{n+1}_{i,p}(x,\theta_0+\omega_i\xi_d)\,\mathrm{d}s+\\
&\sum_{l\neq m, l\neq i, m\neq i} d^i_{l,m}\int_\infty^{\xi_d} \sigma^n_{l,p}(x,\theta_0+\omega_i\xi_d+s(\omega_l-\omega_i))\partial_{\theta_0}\sigma^{n+1}_{m,p}(x,\theta_0+\omega_i\xi_d+s(\omega_m-\omega_i))\,\mathrm{d}s+\\
&\sum_{k\neq i} e^i_k\int_\infty^{\xi_d} \sigma^n_{k,p}(x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))\,\mathrm{d}s-\\
&\sum_{k\neq i}\int_\infty^{\xi_d} V^i_k\sigma^n_{k,p}(x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))\,\mathrm{d}s = \sum_{r=1}^{6} b_{i,r}(x,\theta_0,\xi_d),
\end{aligned}$$

where $b_{i,r}$, $r=1,\ldots,6$ are defined by the respective lines of (4.32). Since $\mathcal{V}^{0,n}$ is bounded in H_T^{s+1}, using Corollary 4.12 we find

$$|b_{i,1}(x,\theta_0,\frac{x_d}{\varepsilon})|_{E_T^{s-2}} \leq C\sum_{k\neq i}\frac{|\sigma^n_k|_{H_T^{s-1}}|\partial_\theta\sigma^{n+1}_k|_{H_T^{s-1}}}{p} \leq C/p\,.$$

Similarly, from Propositions 4.13 and 4.14 we get respectively

$$|b_{i,2}(x,\theta_0,\frac{x_d}{\varepsilon})|_{E_T^{s-2}} \leq C\,,\quad |b_{i,3}(x,\theta_0,\frac{x_d}{\varepsilon})|_{E_T^{s-2}} \leq C/p\,.$$

Since $\mathcal{V}^{0,n}$ is actually bounded in Γ_T^{s+1}, Proposition 4.10 on transversal interactions implies

$$|b_{i,4}(x,\theta_0,\frac{x_d}{\varepsilon})|_{E_T^{s-2}} \leq \frac{C}{p^{t+1}},$$

where we have used Proposition 4.6(b) to estimate

$$|\sigma_{m,p}^{n+1}|_{\Gamma_T^{t+1}} \leq \frac{C}{p^{t+1}}\, |\sigma_m^{n+1}|_{\Gamma_T^{t+1}}\,.$$

By Proposition 4.15 we have $|b_{i,5}(x,\theta_0,\frac{x_d}{\varepsilon})|_{E_T^{s-2}} \leq C$, and the estimate of $b_{i,6}$ is the same, so adding up we obtain

$$|\mathcal{U}_{p,\varepsilon}^1|_{E_T^{s-2}} \leq \frac{C}{p^{t+1}}\,. \tag{4.33}$$

To estimate $(\partial_{x_d}\mathcal{U}_p^1)_\varepsilon$ we differentiate (4.32) and estimate as above to find

$$|(\partial_{x_d}\mathcal{U}_p^1)_\varepsilon|_{E_T^{s-3}} \leq \frac{C}{p^{t+2}}\,. \tag{4.34}$$

9. We claim

$$|\mathbb{L}_0\left(\mathcal{U}_{p,\varepsilon}^{0,n+1} + \varepsilon\,\mathcal{U}_{p,\varepsilon}^1\right) - F(0)\,\mathcal{U}_{p,\varepsilon}^{0,n}|_{E_{T_0}^{s-3}} \leq C\left(\sqrt{p} + \frac{\varepsilon}{p^{t+2}}\right). \tag{4.35}$$

Indeed, we have

$$\mathbb{L}_0(\varepsilon\,\mathcal{U}_{p,\varepsilon}^1) = (\tilde{\mathcal{L}}(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U}_p^1)_\varepsilon + (\tilde{L}(\partial)\varepsilon\,\mathcal{U}_p^1)_\varepsilon + M(\mathcal{U}_{p,\varepsilon}^{0,n},\partial_{\theta_0})(\varepsilon\,\mathcal{U}_{p,\varepsilon}^1),$$

so by (4.24) and (4.30) we find

$$\begin{aligned}\mathbb{L}_0&\left(\mathcal{U}_{p,\varepsilon}^{0,n+1} + \varepsilon\,\mathcal{U}_{p,\varepsilon}^1\right) - F(0)\,\mathcal{U}_{p,\varepsilon}^{0,n} = \\ &\left(\mathbf{E}(\mathcal{G}_p - F(0)\mathcal{U}_p^{0,n})\right)_\varepsilon + D(x,\theta_0,\frac{x_d}{\varepsilon}) + (\tilde{L}(\partial)\varepsilon\,\mathcal{U}_p^1)_\varepsilon + M(\mathcal{U}_{p,\varepsilon}^{0,n},\partial_{\theta_0})(\varepsilon\mathcal{U}_{p,\varepsilon}^1).\end{aligned}$$

The estimate (1.22) now follows from (4.25), (4.31), (4.33), and (4.34).

Using (4.19), (4.22), and (4.35), we obtain

$$\left|\mathbb{L}_0\left(U_\varepsilon^{n+1} - (\mathcal{U}_{p,\varepsilon}^{0,n+1} + \varepsilon\,\mathcal{U}_{p,\varepsilon}^1)\right)\right|_{E_{T_0}^{s-3}} \leq C\,(\sqrt{p} + \varepsilon^a + \frac{\varepsilon}{p^{t+2}})\,. \tag{4.36}$$

10. Next we claim that the following estimates hold:

(4.37)

$$(a)\ \left|\left(\partial_{x_d} + \mathbb{A}(\varepsilon\,\mathcal{U}_{p,\varepsilon}^{0,n},\partial_{x'} + \frac{\beta\,\partial_{\theta_0}}{\varepsilon})\right)\left(U_\varepsilon^{n+1} - (\mathcal{U}_{p,\varepsilon}^{0,n+1} + \varepsilon\,\mathcal{U}_{p,\varepsilon}^1)\right)\right|_{E_{T_0}^{s-3}} \leq C\,(\sqrt{p} + \varepsilon^a + \frac{\varepsilon}{p^{t+2}})\,,$$

$$(b)\ \left|B(\varepsilon\,\mathcal{U}_{p,\varepsilon}^{0,n})\left(U_\varepsilon^{n+1} - (\mathcal{U}_{p,\varepsilon}^{0,n+1} + \varepsilon\,\mathcal{U}_{p,\varepsilon}^1)\right)\right|_{H_{T_0}^{s-2}} \leq C\,(\sqrt{p} + \varepsilon^a + \frac{\varepsilon}{p^{t+2}}).$$

Indeed, (4.37)(a) follows from (4.36) by estimates similar to (4.23), while (4.37)(b) is a simple consequence of (4.14)(b) and (4.15)(c). Applying Proposition 4.17 we find

$$|U_\varepsilon^{n+1} - (\mathcal{U}_{p,\varepsilon}^{0,n+1} + \varepsilon\,\mathcal{U}_{p,\varepsilon}^1)|_{E_{T_0}^{s-3}} \leq C\,\sqrt{T_0}\,(\sqrt{p} + \varepsilon^a + \frac{\varepsilon}{p^{t+2}})\,,$$

and thus

$$|U_\varepsilon^{n+1} - \mathcal{U}_\varepsilon^{0,n+1}|_{E_{T_0}^{s-3}} \leq C\,\sqrt{T_0}\,(\sqrt{p} + \varepsilon^a + \frac{\varepsilon}{p^{t+2}})\,.$$

Recall that t is fixed and equals M_1 in the notation of Theorem 4.16. Setting $p = \varepsilon^b$ we compute $\sqrt{p} = \frac{\varepsilon}{p^{t+2}}$ when $b = \frac{2}{2t+5}$, so we take $a = \frac{b}{2} = \frac{1}{2t+5}$ and complete the induction step by shrinking T_0 to a small enough T_1 if necessary. This completes the proof of Theorem 4.16. $\square$

5 Extension to the general $N \times N$ case

Here we describe the relatively minor changes needed to treat $N \times N$ systems satisfying Assumptions 1.1, 1.2, and 1.6. We first describe the construction of profiles in the general $N \times N$ case. For each $m \in \{1, \ldots, M\}$, let

$$\ell_{m,k}, \; k = 1, \ldots, \nu_{k_m} ,$$

denote a basis of real vectors for the left eigenspace of the real matrix $i\mathcal{A}(\beta)$ associated to the real eigenvalue $-\omega_m$ and chosen to satisfy

$$\ell_{m,k} \cdot r_{m',k'} = \begin{cases} 1, & \text{if } m = m' \text{ and } k = k' , \\ 0, & \text{otherwise.} \end{cases}$$

For $v \in \mathbb{C}^N$ we set

$$P_{m,k} \, v := (\ell_{m,k} \cdot v) \, r_{m,k} \quad \text{(no complex conjugation here).}$$

Functions of type $\mathcal{F}$ (see Definition 1.18) have the form

$$F(x, \theta_0, \xi_d) = \sum_{m=1}^{M} \sum_{k=1}^{\nu_{k_m}} F_{m,k}(x, \theta_0, \xi_d) \, r_{m,k} \tag{5.1}$$

where each scalar function $F_{m,k}$ is decomposed as

$$\begin{aligned} F_{m,k} = \sum_{m'} f^{m,k}_{m'}(x, \theta_0 + \omega_{m'} \, \xi_d) \\ + \sum_{m',k',m'',k''} g^{m,k}_{m',k',m'',k''}(x, \theta_0 + \omega_{m'} \, \xi_d) \, h^{m,k}_{m',k',m'',k''}(x, \theta_0 + \omega_{m''} \, \xi_d) \, . \end{aligned} \tag{5.2}$$

In (5.2), $m' \in \{1, \ldots, M\}$, $k' \in \{1, \ldots, \nu_{k_{m'}}\}$, and similarly for (m'', k''); moreover, the functions $f^{m,k}_{m',k'}$ etc. have the same properties as the corresponding functions in Definition 1.18. The averaging operator $\mathbf{E}$ is given by

$$\mathbf{E}F := \sum_{m,k} \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T F_{m,k}(x, \theta_0 + \omega_m \, (\xi_d - s), s) \, \mathrm{d}s \right) r_{m,k} \, ,$$

and for F as in (5.1), it follows that $\mathbf{E}F = \sum_{m,k} \tilde{F}_{m,k} \, r_{m,k}$ where

$$\tilde{F}_{m,k} := f^{m,k}_{m}(x, \theta_0 + \omega_m \, \xi_d) + \sum_{k',k''} g^{m,k}_{m,k',m,k''}(x, \theta_0 + \omega_m \, \xi_d) \, h^{m,k}_{m,k',m,k''}(x, \theta_0 + \omega_m \, \xi_d).$$

On functions of type $\mathcal{F}$ such that $\mathbf{E}F = 0$, the action of the operator $\mathbf{R}_\infty$ is given by

$$\mathbf{R}_\infty F := \sum_{m,k} \left(\int_\infty^{\xi_d} F_{m,k}(x, \theta_0 + \omega_m(\xi_d - s), s) \, \mathrm{d}s \right) r_{m,k} \, .$$

The general form of the profile equations (1.27) still applies. With

$$W(x, \theta_0, \xi_d) = \sum_{m,k} w_{m,k}(x, \theta_0, \xi_d) \, r_{m,k},$$

the decomposition of Proposition 1.16 now has the form

$$\tilde{L}(\partial)W = \sum_{m,k}(X_{\phi_m} w_{m,k})\, r_{m,k} + \sum_{m,k}\left(\sum_{m'\neq m,k'} V^{m,k}_{m',k'}\, w_{m',k'}\right) r_{m,k},$$

where $V^{m,k}_{m',k'}$ is the tangential vector field

$$V^{m,k}_{m',k'} := \sum_{j=0}^{d-1}(\ell_{m,k}\, \tilde{A}_j(0)\, r_{m',k'})\, \partial_{x_j}.$$

In place of (1.31) and (3.7) we now have

$$\mathcal{U}^{0,n}(x,\theta_0,\xi_d) = \sum_{m=1}^{M}\sum_{k=1}^{\nu_{k_m}} \sigma^n_{m,k}(x,\theta_0+\omega_m\,\xi_d)\, r_{m,k}\,,$$
$$\mathcal{V}^{0,n+1}(x,\theta) = \left(\sigma^{n+1}_{m,k}(x,\theta)\right)_{m=1,\dots,M;\,k=1,\dots,\nu_{k_m}}.$$

The argument that led to the profile system (3.2) now gives[16]

(5.3)

$$(a)\ X_{\phi_m}\sigma^{n+1}_{m,l} + \sum_{j=0}^{d-1}\sum_{k,k'=1}^{\nu_{k_m}} b^{k,k'}_{m,l,j}\,\sigma^n_{m,k}\,\partial_\theta\sigma^{n+1}_{m,k'} = \sum_{k=1}^{\nu_{k_m}} e^k_{m,l}\,\sigma^n_{m,k}\,,$$
$$(b)\ \left(\sigma^{n+1}_{m,k}(x',0,\theta), m\in\mathcal{I}, k=1,\dots\nu_{k_m}\right) = \mathcal{B}\left(G(x',\theta), \sigma^{n+1}_{m,k}(x',0,\theta), m\in\mathcal{O}, k=1,\dots\nu_{k_m}\right),$$
$$(c)\ \sigma^{n+1}_{m,k} = 0 \quad \text{in } t\le 0 \text{ for all } m,k,$$

where the coefficients $b^{k,k'}_{m,l,j}$ are defined by

$$b^{k,k'}_{m,l,j} := \ell_{m,l}\cdot\beta_j\,(\mathrm{d}\tilde{A}_j(0)\, r_{m,k})\, r_{m,k'}. \tag{5.4}$$

Remark 5.1. There is a potentially serious obstacle to proving estimates for the system (5.3). If one takes the L^2 pairing of (5.3)(a) with $\sigma^{n+1}_{m,l}(x,\theta)$, it is not clear how to use integration by parts in θ to move the $\theta-$derivative in the sum on the left onto the n-th iterate. This problem does not arise in the estimate for (3.2). The next Proposition, which is [6, Proposition 2.18], removes this difficulty by showing that there is a symmetry in the coefficients that appears after regrouping.

Definition 5.2. *For u near 0 let $-\omega_m(u)$, $m=1,\dots,M$, be the eigenvalues of*

$$i\,\mathcal{A}(u,\beta) := A_d^{-1}(u)\left(\underline{\tau}\, I + \sum_{j=1}^{d-1}\underline{\eta}_j\, A_j(u)\right),$$

and $P_m(u)$ the corresponding projectors.

The functions $\omega_m(u)$ and $P_m(u)$ are C^∞ for u near 0 since β then belongs to the hyperbolic region of $\mathcal{A}(u,\xi')$.

[16] The nonlinear equations for the functions $\sigma_{m,l}$ are, of course, obtained from (5.3) by removing the superscripts n and $n+1$.

Proposition 5.3. *Let $w \in \mathbb{R}^N$ be expanded as $w = \sum_{m,k} w_{m,k} r_{m,k} = \sum_m w_m$ and define*

$$B^m_{l,k'}(w) := \sum_{j=0}^{d-1} \sum_{k=1}^{\nu_{k_m}} b^{k,k'}_{m,l,j}\, w_{m,k}\,, \tag{5.5}$$

where the $b^{k,k'}_{m,l,j}$ are defined in (5.4)*. Then there holds*

$$B^m_{l,k'}(w) = \begin{cases} -\mathrm{d}\omega_m(0) \cdot w_m & \text{if } k' = l, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Proof. We differentiate the equation

$$\left(\omega_m(u)\, I + \sum_{j=0}^{d-1} \beta_j\, \tilde{A}_j(u) \right) P_m(u) = 0\,,$$

with respect to u in the direction w_m, evaluate at $u = 0$, and apply $P_m := P_m(0)$ on the left to obtain

$$P_m \sum_{j=0}^{d-1} \beta_j \left(\mathrm{d}\tilde{A}_j(0) \cdot w_m \right) P_m = \left(-\mathrm{d}\omega_m(0) \cdot w_m \right) P_m\,. \tag{5.7}$$

The second equality in (5.5) and (5.7) imply (5.6). □

Proposition 5.3 allows us to write

$$\sum_{j=0}^{d-1} \sum_{k,k'=1}^{\nu_{k_m}} b^{k,k'}_{m,l,j}\, \sigma^n_{m,k}\, \partial_\theta \sigma^{n+1}_{m,k'} = B^m_{l,l}(\mathcal{W}^{0,n})\, \partial_\theta \sigma^{n+1}_{m,l}\,, \tag{5.8}$$

where $\mathcal{W}^{0,n} := \sum_{m,k} \sigma^n_{m,k}\, r_{m,k}$; hence we can shift the $\theta-$derivative and integrate by parts as discussed in Remark 5.1. Using (5.8), we deduce from (5.3) that $\sigma^{n+1}_{m,k} = 0$ when $m \in \mathcal{O}$. Otherwise the proof of Proposition 3.5 goes through as before. The statement of Proposition 3.6 is thus unchanged, except that in the second sentence we have $\sigma_{m,k} = 0$ when $m \in \mathcal{O}$ now.

The formulation of Theorem 4.16 is exactly as before except now

$$\mathcal{U}^0(x,\theta_0,\xi_d) = \sum_{m=1}^{M} \sum_{k=1}^{\nu_{k_m}} \sigma_{m,k}(x,\theta_0 + \omega_m \xi_d)\, r_{m,k}\,, \quad \mathcal{V}^0(x,\theta) = \Big(\sigma_{m,k}(x,\theta) \Big)_{m=1,\dots,M;k=1,\dots,\nu_{k_m}}\,.$$

The error analysis in the proof of Theorem 4.16 goes through with the obvious minor changes. For example, the troublesome self-interaction terms $c^i_k\, \sigma^n_{k,p}\, \partial_{\theta_0} \sigma^{n+1}_{k,p}$, $k \neq i$, in (4.27) are now replaced by terms of the form $c^i_{m,k,k'}\, \sigma^n_{m,k,p}\, \partial_{\theta_0} \sigma^{n+1}_{m,k',p}$, $m \neq i$, where the index p as before denotes a moment-zero approximation. These terms are handled just as before by introducing $[(I - \mathbf{E})\, \mathcal{G}]_{mod}$, see (4.28), in which they are replaced by $c^i_{m,k,k'}\, (\sigma^n_{m,k,p}\, \partial_{\theta_0} \sigma^{n+1}_{m,k',p})_p$. The contribution of these terms to the corrector $\mathcal{U}^1_{p,\varepsilon}$ is estimated as before using Corollary 4.12.

A Singular pseudodifferential calculus for pulses

Here we summarize the parts of the singular pulse calculus constructed in [7] that are needed in this article. First we define the singular Sobolev spaces used to describe mapping properties.

The variable in $\mathbb{R}^{d+1}$ is denoted (x,θ), $x \in \mathbb{R}^d$, $\theta \in \mathbb{R}$, and the associated frequency is denoted (ξ,k). In this context, the singular Sobolev spaces are defined as follows. We consider a vector $\beta \in \mathbb{R}^d \setminus \{0\}$. Then for $s \in \mathbb{R}$ and $\varepsilon \in]0,1]$, the anisotropic Sobolev space $H^{s,\varepsilon}(\mathbb{R}^{d+1})$ is defined by

$$H^{s,\varepsilon}(\mathbb{R}^{d+1}) := \Big\{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) \,/\, \widehat{u} \in L^2_{\rm loc}(\mathbb{R}^{d+1}) \\ \text{and} \quad \int_{\mathbb{R}^{d+1}} \left(1 + \left| \xi + \frac{k\,\beta}{\varepsilon} \right|^2 \right)^s \big| \widehat{u}(\xi,k) \big|^2 \, {\rm d}\xi \, {\rm d}k < +\infty \Big\} .$$

Here $\widehat{u}$ denotes the Fourier transform of u on $\mathbb{R}^{d+1}$. The space $H^{s,\varepsilon}(\mathbb{R}^{d+1})$ is equipped with the family of norms

$$\forall \gamma \ge 1 \, , \quad \forall u \in H^{s,\varepsilon}(\mathbb{R}^{d+1}) \, , \quad \|u\|^2_{H^{s,\varepsilon},\gamma} := \frac{1}{(2\,\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \left(\gamma^2 + \left| \xi + \frac{k\,\beta}{\varepsilon} \right|^2 \right)^s \big| \widehat{u}(\xi,k) \big|^2 \, {\rm d}\xi \, {\rm d}k \, .$$

When m is an integer, the space $H^{m,\varepsilon}(\mathbb{R}^{d+1})$ coincides with the space of functions $u \in L^2(\mathbb{R}^{d+1})$ such that the derivatives, in the sense of distributions,

$$\left(\partial_{x_1} + \frac{\beta_1}{\varepsilon} \, \partial_\theta \right)^{\alpha_1} \dots \left(\partial_{x_d} + \frac{\beta_d}{\varepsilon} \, \partial_\theta \right)^{\alpha_d} u \, , \quad \alpha_1 + \cdots + \alpha_d \le m \, ,$$

belong to $L^2(\mathbb{R}^{d+1})$. In the definition of the norm $\| \cdot \|_{H^{m,\varepsilon},\gamma}$, one power of γ counts as much as one derivative.

A.1 Symbols

In this Appendix, $\mathcal{O}$ denotes an open set and nolonger denotes the set of outgoing phases. Our singular symbols are built from the following sets of classical symbols.

Definition A.1. *Let $\mathcal{O} \subset \mathbb{R}^N$ be an open subset that contains the origin. For $m \in \mathbb{R}$, we let $\mathbf{S}^m(\mathcal{O})$ denote the class of all functions $\sigma : \mathcal{O} \times \mathbb{R}^d \times [1,\infty) \to \mathbb{C}^M$, $M \ge 1$, such that σ is C^∞ on $\mathcal{O} \times \mathbb{R}^d$ and for all compact sets $K \subset \mathcal{O}$:*

$$\sup_{v \in K} \, \sup_{\xi \in \mathbb{R}^d} \, \sup_{\gamma \ge 1} \, (\gamma^2 + |\xi|^2)^{-(m-|\nu|)/2} \, |\partial_v^\alpha \partial_\xi^\nu \sigma(v,\xi,\gamma)| \le C_{\alpha,\nu,K} .$$

Let $\mathcal{C}^k_b(\mathbb{R}^{d+1})$, $k \in \mathbb{N}$, denote the space of continuous and bounded functions on $\mathbb{R}^{d+1}$, whose derivatives up to order k are continuous and bounded. Let us next define the singular symbols.

Definition A.2 (Singular symbols)**.** *Fix $\beta \in \mathbb{R}^d \setminus \{0\}$, let $m \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we let S^m_n denote the set of families of functions $(a_{\varepsilon,\gamma})_{\varepsilon \in]0,1], \gamma \ge 1}$ that are constructed as follows:*

$$\text{(A.1)} \qquad \forall \, (x,\theta,\xi,k) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \, , \quad a_{\varepsilon,\gamma}(x,\theta,\xi,k) = \sigma \left(\varepsilon \, V(x,\theta), \xi + \frac{k\,\beta}{\varepsilon}, \gamma \right) ,$$

where $\sigma \in \mathbf{S}^m(\mathcal{O})$, V belongs to the space $\mathcal{C}^n_b(\mathbb{R}^{d+1})$ and where furthermore V takes its values in a convex compact subset K of $\mathcal{O}$ that contains the origin (for instance K can be a closed ball centered round the origin).

All results below extend to the case where in place of a function V that is independent of ε, the representation (A.1) is considered with a function V_ε that is indexed by ε, provided that we assume that all functions $\varepsilon\, V_\varepsilon$ take values in a *fixed* convex compact subset K of $\mathcal{O}$ that contains the origin, and $(V_\varepsilon)_{\varepsilon\in(0,1]}$ is a bounded family of $\mathcal{C}_b^n(\mathbb{R}^{d+1})$. Such singular symbols with a function V_ε are exactly the kind of symbols that we manipulated in the construction of exact solutions to the singular system (1.3).

A.2 Definition of operators and action on Sobolev spaces

To each symbol $a = (a_{\varepsilon,\gamma})_{\varepsilon\in]0,1],\gamma\geq 1} \in S_n^m$ given by the formula (A.1) and with values in $\mathbb{C}^{N\times N}$, we associate a singular pseudodifferential operator $\mathrm{Op}^{\varepsilon,\gamma}(a)$, with $\varepsilon \in]0,1]$ and $\gamma \geq 1$, whose action on a function $u \in \mathcal{S}(\mathbb{R}^{d+1};\mathbb{C}^N)$ is defined by

$$\text{(A.2)} \qquad \mathrm{Op}^{\varepsilon,\gamma}(a)\, u\,(x,\theta) := \frac{1}{(2\,\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \mathrm{e}^{i\,(\xi\cdot x + k\,\theta)}\, \sigma\left(\varepsilon\, V(x,\theta), \xi + \frac{k\,\beta}{\varepsilon}, \gamma\right) \widehat{u}(\xi,k)\, \mathrm{d}\xi\, \mathrm{d}k\,.$$

Let us briefly note that for the Fourier multiplier $\sigma(v,\xi,\gamma) = i\,\xi_1$, the corresponding singular operator is $\partial_{x_1} + (\beta_1/\varepsilon)\,\partial_\theta$. We now describe the action of singular pseudodifferential operators on Sobolev spaces.

Proposition A.3. *Let $n \geq d+1$, and let $a \in S_n^m$ with $m \leq 0$. Then $\mathrm{Op}^{\varepsilon,\gamma}(a)$ in* (A.2) *defines a bounded operator on $L^2(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation* (A.1)*, such that for all $\varepsilon \in]0,1]$ and for all $\gamma \geq 1$, there holds*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|\mathrm{Op}^{\varepsilon,\gamma}(a)\, u\|_0 \leq \frac{C}{\gamma^{|m|}}\, \|u\|_0\,.$$

The constant C in Proposition A.3 depends uniformly on the compact set in which V takes its values and on the norm of V in $\mathcal{C}_b^{d+1}$. For operators defined by symbols of order $m > 0$ we have:

Proposition A.4. *Let $n \geq d+1$, and let $a \in S_n^m$ with $m > 0$. Then $\mathrm{Op}^{\varepsilon,\gamma}(a)$ in* (A.2) *defines a bounded operator from $H^{m,\varepsilon}(\mathbb{R}^{d+1})$ to $L^2(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation* (A.1)*, such that for all $\varepsilon \in]0,1]$ and for all $\gamma \geq 1$, there holds*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|\mathrm{Op}^{\varepsilon,\gamma}(a)\, u\|_0 \leq C\, \|u\|_{H^{m,\varepsilon},\gamma}\,.$$

The next proposition describes the smoothing effect of operators of order -1.

Proposition A.5. *Let $n \geq d+2$, and let $a \in S_n^{-1}$. Then $\mathrm{Op}^{\varepsilon,\gamma}(a)$ in* (A.2) *defines a bounded operator from $L^2(\mathbb{R}^{d+1})$ to $H^{1,\varepsilon}(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation* (A.1)*, such that for all $\varepsilon \in]0,1]$ and for all $\gamma \geq 1$, there holds*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|\mathrm{Op}^{\varepsilon,\gamma}(a)\, u\|_{H^{1,\varepsilon},\gamma} \leq C\, \|u\|_0\,.$$

Remark A.6. In applications of the pulse calculus, we verify the hypothesis that for V as in (A.1), $V \in \mathcal{C}_b^n(\mathbb{R}^{d+1})$, by showing $V \in H^s(\mathbb{R}^{d+1})$ for some $s > \frac{d+1}{2} + n$.

A.3 Adjoints and products

For proofs of the following results we refer to [7]. The two first results deal with adjoints of singular pseudodifferential operators while the last two deal with products.

Proposition A.7. *Let $a = \sigma(\varepsilon V, \xi + \frac{k\,\beta}{\varepsilon}, \gamma) \in S^0_n$, $n \geq 2\,(d+1)$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$, and let a^* denote the conjugate transpose of the symbol a. Then $\mathrm{Op}^{\varepsilon,\gamma}(a)$ and $\mathrm{Op}^{\varepsilon,\gamma}(a^*)$ act boundedly on L^2 and there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0,1]$ and for all $\gamma \geq 1$, there holds*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|\mathrm{Op}^{\varepsilon,\gamma}(a)^*\, u - \mathrm{Op}^{\varepsilon,\gamma}(a^*)\, u\|_0 \leq \frac{C}{\gamma}\, \|u\|_0\,.$$

If $n \geq 3\,d+3$, then for another constant C, there holds

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|\mathrm{Op}^{\varepsilon,\gamma}(a)^*\, u - \mathrm{Op}^{\varepsilon,\gamma}(a^*)\, u\|_{H^{1,\varepsilon},\gamma} \leq C\, \|u\|_0\,,$$

uniformly in ε and γ.

Proposition A.8. *Let $a = \sigma(\varepsilon V, \xi + \frac{k\,\beta}{\varepsilon}, \gamma) \in S^1_n$, $n \geq 3\,d+4$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$, and let a^* denote the conjugate transpose of the symbol a. Then $\mathrm{Op}^{\varepsilon,\gamma}(a)$ and $\mathrm{Op}^{\varepsilon,\gamma}(a^*)$ map $H^{1,\varepsilon}$ into L^2 and there exists a family of operators $R^{\varepsilon,\gamma}$ that satisfies*

- *there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0,1]$ and for all $\gamma \geq 1$, there holds*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|R^{\varepsilon,\gamma}\, u\|_0 \leq C\, \|u\|_0\,,$$

- *the following duality property holds*

$$\forall\, u,v \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \langle \mathrm{Op}^{\varepsilon,\gamma}(a)\, u, v\rangle_{L^2} - \langle u, \mathrm{Op}^{\varepsilon,\gamma}(a^*)\, v\rangle_{L^2} = \langle R^{\varepsilon,\gamma}\, u, v\rangle_{L^2}\,.$$

In particular, the adjoint $\mathrm{Op}^{\varepsilon,\gamma}(a)^*$ for the L^2 scalar product maps $H^{1,\varepsilon}$ into L^2.

Proposition A.9. *(a) Let $a, b \in S^0_n$, $n \geq 2\,(d+1)$, and suppose $b = \sigma(\varepsilon V, \xi + \frac{k\,\beta}{\varepsilon}, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0,1]$ and for all $\gamma \geq 1$, there holds*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|\mathrm{Op}^{\varepsilon,\gamma}(a)\, \mathrm{Op}^{\varepsilon,\gamma}(b)\, u - \mathrm{Op}^{\varepsilon,\gamma}(a\, b)\, u\|_0 \leq \frac{C}{\gamma}\, \|u\|_0\,.$$

If $n \geq 3\,d+3$, then for another constant C, there holds

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|\mathrm{Op}^{\varepsilon,\gamma}(a)\, \mathrm{Op}^{\varepsilon,\gamma}(b)\, u - \mathrm{Op}^{\varepsilon,\gamma}(a\, b)\, u\|_{H^{1,\varepsilon},\gamma} \leq C\, \|u\|_0\,,$$

uniformly in ε and γ.

(b) Let $a \in S^1_n, b \in S^0_n$ or $a \in S^0_n, b \in S^1_n$, $n \geq 3\,d+4$, and in each case suppose $b = \sigma(\varepsilon V, \xi + \frac{k\,\beta}{\varepsilon}, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0,1]$ and for all $\gamma \geq 1$, there holds

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad \|\mathrm{Op}^{\varepsilon,\gamma}(a)\, \mathrm{Op}^{\varepsilon,\gamma}(b)\, u - \mathrm{Op}^{\varepsilon,\gamma}(a\, b)\, u\|_0 \leq C\, \|u\|_0\,.$$

Our final result is Gårding's inequality.

Theorem A.1. *Let $\sigma \in \mathbf{S}^0$ satisfy $\mathrm{Re}\, \sigma(v, \xi, \gamma) \geq C_K > 0$ for all v in a compact subset K of $\mathcal{O}$. Let now $a \in S^n_0$, $n \geq 2\,d+2$ be given by* (A.1)*, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$ and is valued in a convex compact subset K. Then for all $\delta > 0$, there exists γ_0 which depends uniformly on V, the constant C_K and δ, such that for all $\gamma \geq \gamma_0$ and all $u \in \mathcal{S}(\mathbb{R}^{d+1})$, there holds*

$$\mathrm{Re}\, \langle \mathrm{Op}^{\varepsilon,\gamma}(a)\, u; u\rangle_{L^2} \geq (C_K - \delta)\, \|u\|_0^2\,.$$

A.4 Extended calculus

In our proof of $L^\infty(x_d; L^2(x', \theta_0))$ estimates for the linearized singular system (Theorem 2.3), we use a slight extension of the singular calculus. For given parameters $0 < \delta_1 < \delta_2 < 1$, we choose a cutoff $\chi^e(\xi', \frac{k\,\beta}{\varepsilon}, \gamma)$ such that

$$\begin{aligned} &0 \le \chi^e \le 1\,, \\ &\chi^e\left(\xi', \frac{k\,\beta}{\varepsilon}, \gamma\right) = 1 \text{ on } \left\{(\gamma^2 + |\xi'|^2)^{1/2} \le \delta_1 \left|\frac{k\,\beta}{\varepsilon}\right|\right\}\,, \\ &\operatorname{supp} \chi^e \subset \left\{(\gamma^2 + |\xi'|^2)^{1/2} \le \delta_2 \left|\frac{k\,\beta}{\varepsilon}\right|\right\}\,, \end{aligned} \tag{A.3}$$

and define a corresponding Fourier multiplier χ^e_D in the extended calculus by the formula (A.2) with $\chi^e(\xi', \frac{k\,\beta}{\varepsilon}, \gamma)$ in place of $\sigma(\varepsilon V, X, \gamma)$. Composition laws involving such operators are proved in [7], but here we need only the fact that part *(a)* of Proposition A.9 holds when either a or b is replaced by an extended cutoff χ^e.

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